SECOND COURSE IN ALGEBRA

BY

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PREFACE

THE Report of the National Committee on Mathematical Requirements, the new requirements of the College Entrance Examination Board, and the many researches in pedagogy have changed the mathematical work done in the first course and necessitated a revision of the subsequent work in algebra.

This Second Course in Algebra (1) furnishes a thorough review and drill of the topics that are no longer extensively treated in the first course, and (2) presents the applications to statistics, geometry, logarithms, numerical trigonometry, and science that are required in a modern course.

Attention is called to the following points:

1. The first chapter is a review and summary of the fundamental operations, and the first one hundred pages are a review of the fundamental topics. A few new types of factoring and a greater variety of problems are introduced where they are extensions of the previous work.

2. At the beginning of each chapter, the fundamental ideas are explained with as much detail as in any first course in algebra. Thus, even the beginnings of quadratic equations are treated fully before the subject advances to equations reducible to the quadratic type and to literal quadratic equations.

3. The division of each chapter into small units, each with a clear title, enables the teacher to select easily the particular topics needed by the class or by individual pupils for review while advancing in the work.

4. After presenting a review of the fundamentals, the text does not attempt to include every topic that might be studied or every type of exercise that ingenuity can invent. The amount of space devoted to each subject is proportional to its importance or its interest and is planned so that the work can be covered in half a year.

5. The material in the book is so arranged that each explanation is begun and completed on a single page. The accompanying exercises are either on the same page or on the right-hand page, *facing the explanation.* This arrangement encourages the pupil, when working the exercises, to refer frequently to the explanation and to the examples in the book. The arrangement makes a unit of the explanation of the topic and the problems dealing with that topic and therefore makes it easier for the teacher to plan and to assign the work.

6. The chapter on logarithms may be studied at any time after the chapter on exponents, because the problems involving progressions or compound interest are not mixed in with the usual applications of logarithms to computations and formulas, but are treated as a separate topic.

There are occasional discussions, like those on pages 104, 120, 165, 188, and 261, that belong in a textbook although they may never be assigned for study. If such topics are read in class with comments by the teacher, they will serve to indicate to the pupil the nature of other fields of mathematics and will awaken an interest in the subject.

CONTENTS

COURSES COVERED IN THIS BOOK

1. A MINIMUM COURSE

Pages 7 to 237 include what most teachers would consider a minimum course. In these pages the supplementary work (18 pages in all) is not considered as part of the minimum course. Topics from the first course in algebra are included for convenience in reviewing.

2. A COURSE SUGGESTED BY THE RECOMMENDATIONS OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS

The National Committee recommends the inclusion of a proof of the binomial theorem for positive integral exponents and the applications of the binomial theorem to negative and fractional exponents (pages 217 to 219).

3. A COURSE SUGGESTED BY THE REQUIREMENTS OF THE COLLEGE ENTRANCE EXAMINATION BOARD

In addition to the minimum course, the College Entrance Examination Board requires a study of the complete solution of a right triangle (pages 240 to 257) in which the given parts are measured to four significant figures. Other specific requirements have been met by special exercises in various places (certain sets of quadratic equations, page 180; factoring of $a^n \pm b^n$ as a topic under progressions, page 207; and applications of the binomial theorem to compound interest, page 216).

SECOND COURSE IN ALGEBRA

CHAPTER I

REVIEW OF FUNDAMENTAL OPERATIONS

1. In earlier work we learned that algebra differs from arithmetic particularly in three ways:

1. Algebra uses negative as well as positive numbers. Thus, temperatures above zero are written as $+5$ and $+10$, while those below zero are written as -5 and -10. When speaking, we usually say "plus 5" instead of "positive 5" for $+5$; likewise we say "minus 5" instead of "negative 5" for -5.

2. Algebra uses letters of the alphabet to represent numbers. The number of feet in the length of a rectangle, for example, is represented by the letter l, the number of feet in the width by w, and the number of square feet in the area by A. When letters are thus used to represent numbers, we call them *literal numbers*.

3. Algebra uses equations to express some relation between numbers or to state concisely how one number depends on other numbers. Thus, $y = x + 2$ is a brief way of stating that one number, y, is 2 more than another number, x; and $A = lw$ is a concise way of stating how the area of a rectangle depends on its length and width.

Since all mathematics uses four fundamental operations — addition, subtraction, multiplication, and division — we shall first review these topics from the earlier work in algebra.

2. Addition of Positive and Negative Numbers. A man gains $10 in one transaction and loses $14 in the next. What is the final result of the two transactions?

We say that $+10$ is the result of the first and -14 the result of the second. Since we know that the final result is a loss of $4, we see that the sum of $+10$ and -14 is -4. This is written $+10-14=-4$.

There is no sign for addition in this statement. To show that we wish to add a list of numbers, we merely write them one after the other. Since the signs $+$ and $-$ separate the individual numbers 10 and 14, we cannot mistake these numbers for the single number 1014. We can likewise see that $+10-7=+3$, $-10+7=-3$, $-10-7=-17$.

To be able to state briefly the rules of addition, we introduce the words "absolute value of a number" to mean the number that we get when the sign is disregarded. For example, the absolute value of $+5$ is 5, and of -5 is also 5. We can then state the rules:

The sum of two positive numbers is a positive number and is found by adding the absolute values of the two numbers.

The sum of two negative numbers is a negative number and is found by adding the absolute values of the two numbers.

The sum of a positive number and a negative number may be either a positive number or a negative number and is found as follows:

Find the difference between the absolute values of the numbers and then prefix the sign $+$ or $-$ to the result, according to whether the positive number or the negative number has the greater absolute value.

The diagram at the bottom of page 9 may be used for oral practice. Add each number in any row or column to the corresponding number in any other row or column; also find the sum of each column and each row.

3. Multiplication of Positive and Negative Numbers. Consider the four products:

(*a*) $+5 \cdot +6 = +30$ because $+5 \cdot +6$ means $+6+6+6+6+6$ which equals $+30$.

(*b*) $-5 \cdot +6 = -30$ because -5 times any number must be the opposite of $+5$ times that number.

(*c*) $+5 \cdot -6 = -30$ because $+5 \cdot -6$ means $-6-6-6-6-6$ which equals -30.

(*d*) $-5 \cdot -6 = +30$ because -5 times any number must be the opposite of $+5$ times that number.

Read and complete the following statements:

The product of two numbers having like signs is $\cdots$

The product of two numbers having unlike signs is $\cdots$

Numbers that are added are called *terms*. Numbers that are multiplied are called *factors*. The sum of several terms does not depend on the order in which we add the terms. Thus:

$$-6+5+8=+5-6+8=+5+8-6=7$$

Neither does the product depend on the order in which we multiply the factors. Thus:

$$-4 \cdot +5 \cdot -6 = -4 \cdot -6 \cdot +5 = +5 \cdot -4 \cdot -6 = 120$$

The following diagram may be used for practice. Find the product of each number in any row or column by the corresponding number in any other row or column; also find the product of all the numbers in each column, etc.

	I	II	III	IV	V	VI	VII	VIII
A	-2	0	-3	$+6$	-4	-5	$\frac{1}{2}$	$-.8$
B	-4	$+8$	$+4$	-9	1	-6	$-\frac{1}{4}$	$+.5$
C	-9	$+1$	0	$+5$	-7	-8	$\frac{2}{3}$	1.2
D	$+3$	-5	$+3$	-2	$+7$	$+9$	$\frac{2}{5}$	$-.6$

4. Division of Positive and Negative Numbers. We wish to find the values of the following fractions:

$$\frac{+30}{+5} = ? \qquad \frac{+30}{-5} = ? \qquad \frac{-30}{+5} = ? \qquad \frac{-30}{-5} = ?$$

Evidently the quotient in each case is either $+6$ or -6. Find which is correct by multiplying the divisor by the quotient.

Write a rule for telling when a quotient should be a positive number, and when it should be a negative number.

Complete the sentences:

The quotient of two numbers with like signs is · · ·

The quotient of two numbers with unlike signs is · · ·

How do these rules compare with the rules for multiplication found on page 9?

Since $5 \cdot 0 = 0$ and $6 \cdot 0 = 0$, it follows that $5 \cdot 0 = 6 \cdot 0$. If we could divide both members of this equation by 0, we should get $5 = 6$, which is evidently not true. This illustrates the following important idea:

The number 0 *may not be used as a divisor.*

5. ORAL EXERCISES

1. Divide each of the following numbers:

-30	60	-90	-60	15	48

by (*a*) -3 (*c*) -6 (*e*) -10 (*g*) $+1$
(*b*) $+6$ (*d*) -2 (*f*) -5 (*h*) -1

2. What effect on a number does division by $+1$ have? What effect on a number does division by -1 have? What effect does multiplication by -1 have?

State the indicated quotients:

3. $\dfrac{-2.4}{.8}$ 5. $\dfrac{-.32}{.08}$ 7. $\dfrac{15}{-.5}$ 9. $\dfrac{-.15}{.05}$

4. $\dfrac{3.2}{-.8}$ 6. $\dfrac{-3.2}{.08}$ 8. $\dfrac{.15}{-.5}$ 10. $\dfrac{-1.5}{-.05}$

6. Evaluation of Algebraic Expressions. If a and b represent two numbers, then an expression like $3a + b^2$ also represents a number which we call the numerical value of the quantity. To find its value we must understand the symbols by which algebra indicates sums, products, quotients, etc. Thus:

$3a$ means the product of the numbers 3 and a.

$-3a$ means the product of the numbers -3 and a.

$-a$ means -1 times a; that is, the opposite of a.

b^2 means $b \cdot b$; b^3 means $b \cdot b \cdot b$; etc.

ab means the product of the numbers a and b.

$-6xy$ means that the product xy is multiplied by -6.

$b^2 - 3a$ means the sum of the numbers b^2 and $-3a$; this may also be thought of as the subtraction of $3a$ from b^2.

$\frac{5c}{3a+4b}$ means that the value of $5c$ is divided by the value of the quantity $3a + 4b$.

Example. If $a = 3$, $b = 4$, $c = -5$, find the value of the expression $\frac{2a^2 - 3b - 6c}{5b + c}$.

The numerator contains three terms, $2a^2$, $-3b$, and $-6c$. If $a = 3$, then $a^2 = 9$ and $2a^2 = 18$. Also $-3b = -12$ and $-6c = +30$. In the denominator, $5b = 20$ and $c = -5$. Get the habit of evaluating each term mentally and writing down one term at a time. The written work should be:

Given: $a = 3$, $b = 4$, $c = -5$

Then $\frac{2a^2 - 3b - 6c}{5b + c} = \frac{18 - 12 + 30}{20 - 5} = \frac{36}{15} = 2\frac{2}{5}$

If you have difficulty with the mental work, use parentheses, ( ), to show the multiplications. Thus:

$$\frac{2a^2 - 3b - 6c}{5b + c} = \frac{2(9) - 3(4) - 6(-5)}{5(4) - 5} = \frac{18 - 12 + 30}{20 - 5} = \text{etc.}$$

7. EXERCISES

Do ex. 1 to 10 orally.

1. If $x = 2$, state the values of:

$x^2, \quad 3x^2, \quad -3x^2, \quad 5x^2, \quad -5x^2, \quad -x^2$

2. Do ex. 1 again, using (a) $x = -2$; (b) $x = \frac{1}{2}$.

3. If $a = 3$, state the values of:

$a^3, \quad 2a^3, \quad -2a^3, \quad 5a^3, \quad -5a^3, \quad -a^3$

4. Do ex. 3 again, using (a) $a = -3$; (b) $a = \frac{1}{3}$.

5. If $b = .5$, state the values of:

$b^2, \quad -b^2, \quad b^3, \quad -b^3, \quad 2b^2, \quad 2b^3$

6. Do ex. 5 again, using (a) $b = .4$; (b) $b = -.4$.

State the values of the following quantities, given that $a = 2$, $b = -3$, $c = \frac{1}{2}$, $d = .4$:

7. $a^2, \quad ab, \quad 4ab, \quad -4ab, \quad -ab$

8. $b^2, \quad bd, \quad -2bd, \quad bc, \quad -bc$

9. $c^2, \quad c^3, \quad ac, \quad a^2c, \quad ac^2$

10. $d^2, \quad d^3, \quad ad, \quad ad^2, \quad a^2d$

If $r = 3$, $s = 4$, $t = -5$, find the values of the following expressions, writing the work as shown on page 11:

11. $\dfrac{4r^2 + st}{5}$ **13.** $\dfrac{r^2 + s^2}{t^2}$ **15.** $\dfrac{r^2 + s^2 + t^2}{r + s + t}$

12. $\dfrac{s^2 + rt}{10}$ **14.** $\dfrac{s^2 - 2t^2}{r^2 + 2s}$ **16.** $\dfrac{r^3 + s^3 + t^3}{2r^2 + s^2}$

17. If $x = 5$, $y = 4$, $z = -3$:

(a) What is the value of $3x + 2y + 5z$?

(b) What is the value of $xy - 20$?

(c) Does $3x + 2y + 5z$ equal $xy - 20$?

(d) Are *both* of the following equations true?

$$3x + 2y - z = 26$$
$$4x - 2y + z = 10$$

ANSWERS: **11.** $3\frac{1}{5}$. **12.** $\frac{1}{10}$. **13.** 1. **14.** -2. **15.** 25. **16.** -1.

8. Addition of Similar Terms. Similar terms, such as $6x$, $-3x$, and $+8x$, can be added. If the question is asked, "By what numbers is x multiplied in the expression $6x - 3x + 8x$?" the answer is "By $6 - 3 + 8$; that is. by 11." Hence $6x - 3x + 8x = 11x$.

A polynomial, such as $9x^2 - 8x - 4x^2 + 2x$, can be simplified whenever some of its terms can be added. In this polynomial, $9x^2 - 4x^2 = 5x^2$, and $-8x + 2x = -6x$. Hence we can write $9x^2 - 8x - 4x^2 + 2x = 5x^2 - 6x$.

We cannot, however, actually add $5x^2$ and $-6x$ because x^2 and x are not the same number (unless $x = 1$ or 0).

9. ORAL EXERCISES

By adding the similar terms, simplify:

1. $3x^2 - 5x - 2x^2 + 3x$
2. $2y^2 - 3 + 6y^2 + 4$
3. $3a - 2b - 4a + 2b$
4. $a^3 - ab - 3a^3 - ab$
5. $4b^3 + b - 2b + b^3$
6. $1.2x + .7 + 9.3x - .8$
7. $.6y + .3z - 3.2z - .4y$
8. $1.5a - .4b + b - 2a$
9. $.8r - .9s + s - .9r$
10. $.6c + 1.4d - .8c - d$

11.	12.	13.
$5x - 7y - 3z$	$2a + 3b - 5c$	$5r + 2s + t$
$-2x + 6y - z$	$-a + b + 7c$	$2r - 2t$
$x - 2y + 5z$	$a - 4b - 3c$	$r - 3s$

Even when the similar terms are not written in columns, the polynomials can be added by adding first all the terms that contain x, then the terms that contain y, etc.

Add:

14. $3a + 2b - 5c,\ -2a + 4b + 9c,\ 4a - 3b$
15. $4x^2 - 3x + 2,\ -2x + 5,\ -x^2 + 5x - 12$
16. $2xy + 3x - 5y,\ -3xy - 8x,\ 6xy + 9y$
17. $\frac{1}{3}a + \frac{3}{4}b - c,\ \frac{1}{2}a + \frac{1}{4}b - \frac{2}{5}c$
18. $\frac{1}{5}x - \frac{1}{2}y + z,\ \frac{1}{2}x + \frac{1}{3}y - \frac{3}{4}z$
19. $\frac{3}{4}r + s + \frac{1}{5},\ \frac{1}{2}r - \frac{1}{4}s - \frac{1}{10},\ r - \frac{1}{2}s - \frac{1}{10}$
20. $\frac{1}{6}u + \frac{1}{5}v - \frac{1}{8},\ \frac{1}{4}u - \frac{3}{10}v + \frac{3}{4},\ \frac{1}{3}u + v + \frac{1}{2}$

10. Subtraction. In subtracting 5 from 8, we really find what number added to 5 makes 8. Again, in subtracting -12 from $+7$, we find what number added to -12 makes $+7$. To find the answer quickly, we may use the rule:

To subtract a number, change its sign (mentally) and then follow the rules for adding numbers. Thus:

Minuend	-22	-22	-10	$+10$	$2a-3b+5c$
Subtrahend	-10	$+10$	-22	-22	$5a+4b-3c$
Difference	-12	-32	$+12$	$+32$	$-3a-7b+8c$

11. ORAL EXERCISES

1. From each of the following numbers

-16, 18, 25, -15, -10, $+4$, -6

subtract (*a*) -9 (*c*) $+20$ (*e*) -12 (*g*) -3
(*b*) -10 (*d*) $+6$ (*f*) -40 (*h*) $+\frac{1}{2}$

The diagram on page 9 may be used for further practice.

Subtract the lower polynomial from the upper:

2.
$$\begin{array}{r} 3a-7b+4c \\ \underline{4a-2b-3c} \end{array}$$

4.
$$\begin{array}{r} 3a^2+a-4 \\ \underline{8a^2-a+3} \end{array}$$

6.
$$\begin{array}{r} x-y+\frac{1}{2}z \\ \underline{x+\frac{1}{4}y+\frac{1}{3}z} \end{array}$$

3.
$$\begin{array}{r} a-2b+3c \\ \underline{8a-b-2c} \end{array}$$

5.
$$\begin{array}{r} 6a^2-2a+5 \\ \underline{6a^2+7a+5} \end{array}$$

7.
$$\begin{array}{r} .2x+.5y-z \\ \underline{.8x-y+.3z} \end{array}$$

8. From $8a+3b$ subtract $-2a-b+3c$.
9. Subtract $2x^2-5x+7$ from $-9x^2+2x-3$.
10. Subtract $3x-2y+7z$ from $x-2y+3z$.
11. From $4a-3b+2c$ subtract $a-5b-c$.
12. How much larger than $-2x+3$ is $5x+9$?
13. How much smaller than $2b+3c$ is $5a-.7b$?
14. How much smaller is $\frac{1}{2}a-b$ than $\frac{2}{3}a+b$?
15. How much larger is $.5x-.8$ than $.9x+.3$?
16. $3x^2-2x$ is how much larger than $7x^2-6x$?

15. Multiplication of Polynomials by Monomials. To multiply a polynomial, such as $2x^2 - 3xy + 4y^2$, by any monomial, such as $5x$, multiply each term of the polynomial by the monomial; that is, multiply the first term, $2x^2$, by $5x$; then multiply the second term, $-3xy$, by $5x$; then multiply the next term, $+4y^2$, by $5x$. Do this work mentally, merely writing

$$5x(2x^2 - 3xy + 4y^2) = 10x^3 - 15x^2y + 20xy^2$$

16. ORAL EXERCISES

State the indicated products:

1. $2x(5x^2 - 3x + 4)$
2. $-3x(5x^2 - 2x + 4)$
3. $3xy(2x^2 + 3x + 5)$
4. $-3x^2(4x^2 - 9x - 4)$
5. $3x^2(4x^2 - 5xy - 3y^2)$
6. $4x^2(5x^2y - 3xy^2)$
7. $-5(a^2 - 2ab + b^2)$
8. $-2a(a^2 - b^2 + c^2)$
9. $x^2(2x^2 - 3xy + 4)$
10. $5x(x^3 - 3x^2 - 2x)$
11. $-3y(1 - 2y + 3y^2)$
12. $x^3(1 + x + x^2)$
13. $a(a^2 - 3a + 5)$
14. $-a(a^2 + 4a - 6)$
15. $-r(r^2 + 3r - 5)$
16. $3b(b^3 + b^2 - b)$
17. $-3ab(a^3 - ab + b^3)$
18. $a(ab - bc + ac)$
19. $-a(a + b - c)$
20. $-a^2(a - ab + b)$
21. $x(x + xy + 1)$
22. $ab(a - b + c)$
23. $20\left(\frac{y}{5} - \frac{3}{4}\right)$
24. $10\left(\frac{x}{10} - \frac{3y}{5}\right)$
25. $30\left(\frac{2x}{5} - 2\right)$
26. $24\left(\frac{5a}{8} + \frac{2}{3}\right)$
27. $40\,\frac{(7 - 3x)}{8}$
28. $-15\,\frac{(x - 2)}{5}$
29. $\frac{1}{3}(6a - 15b)$
30. $\frac{5}{6}(12r - 6s)$
31. $\frac{2}{3}(6x - 15)$
32. $\frac{3}{4}(8x - 20)$
33. $.5(.3a - 1.6b)$
34. $-.5(1.2r - 2.5s)$
35. $.04(6x - 2y - .1z)$
36. $-1.2(.3a - .8b)$
37. $.75(.2x - .1y)$
38. $.2(.3x - .03y)$

17. Multiplication of Polynomials. To multiply two quantities, such as $2a^2 - 5ab - b^2$ by $4a - 3b$, we multiply the polynomial first by $4a$ and then by $-3b$ and combine the partial products. The written work is arranged thus:

$$
\begin{array}{lrrrrl}
 & 2a^2 & -\ 5ab & -\ b^2 & & \\
 & 4a & -\ 3b & & & \\
\hline
\text{Multiply by } 4a: & 8a^3 & -\ 20a^2b & -\ 4ab^2 & & \text{1st Partial Product} \\
\text{Multiply by } -3b: & & -\ 6a^2b & +\ 15ab^2 & +\ 3b^3 & \text{2d Partial Product} \\
\hline
 & 8a^3 & -\ 26a^2b & +\ 11ab^2 & +\ 3b^3 & \text{Complete Product}
\end{array}
$$

The term $-6a^2b$ in the second partial product is written below the term $-20a^2b$ in the first partial product because these two terms are alike and can be added. In some exercises only some of the terms are alike, and many columns will be needed to show the partial products. Thus:

$$
\begin{array}{rrrrr}
2x & -\ y & +\ 4z & & \\
3x & -\ 5z & & & \\
\hline
6x^2 & -\ 3xy & +\ 12xz & & \\
 & & -\ 10xz & +\ 5yz & -\ 20z^2 \\
\hline
6x^2 & -\ 3xy & +\ 2xz & +\ 5yz & -\ 20z^2
\end{array}
$$

To check the result of any multiplication, substitute numerical values for the literal numbers. The value of the product should be the product of the values of the two factors. In the example below, the values $x = 2$, $y = 3$ are used:

$$
\begin{array}{llll}
x^2 + xy - 2y^2 & = & 4 + 6 - 18 & = -8 \\
4x - y & = & 8 - 3 & = 5 \\
\hline
4x^3 + 4x^2y - 8xy^2 & & & -40 \\
\quad - x^2y - xy^2 + 2y^3 & & & \\
\hline
4x^3 + 3x^2y - 9xy^2 + 2y^3 & = & 32 + 36 - 162 + 54 & = -40
\end{array}
$$

Any convenient numbers may be used for checking; but if the same number is used for both letters or if the number 1 is used, we are not so certain of discovering an error.

18. EXERCISES

Find the indicated products:

1. $(3x - 2)(x^2 - 4x + 5)$
2. $(2y + 5)(y^2 - 3y + 1)$
3. $(4a - 3)(a^2 - 2a + 5)$
4. $(5r - 1)(5r^2 + r - 3)$
5. $(2s + 3)(2s^2 - 3s + 1)$
6. $(2x + 1)(x^3 - x^2 + 4)$
7. $(3y - 2)(2y^3 - 6y + 1)$
8. $(4b - 1)(4b^3 - 3b^2 + 2)$
9. $(b + 4)(6b^4 - 2b^3 + 3)$
10. $(3r^2 - 1)(r^4 - r^2 + 1)$
11. $(3x - 2)(4x^2 - 5 + 6x)$ Here $4x^2 - 5 + 6x$ should first be written as $4x^2 + 6x - 5$. This rearrangement is not absolutely necessary, but it will prevent many errors. What idea is used in arranging the terms?
12. $(5x - 2y)(3x^2 + y^2 - 4xy)$
13. $(a - 3b)(9b^2 + a^2 + 3ab)$
14. $(a - 2b + 3c)(a^2 + 2ab + ac - 6bc)$
15. $(a + x + y)(a - x - y) + (a - x + y)(a + x - y)$
16. $\left(\frac{a^2}{2} - \frac{a}{3} + \frac{1}{4}\right)\left(\frac{a^2}{6} + \frac{a}{2} - \frac{1}{3}\right)$ Here we might be tempted to avoid fractions by multiplying the first quantity by 12 and the second by 6 and then multiplying the results; but this would make the product 12×6, or 72, times larger than it should be. Either the quantities should be multiplied as they are, or, if multipliers like 12 and 6 are used, the final result should be divided by the product 12×6.

If the same multiplier, 12, is used for both quantities, by what should the result be divided?

17. $\left(\frac{b^2}{3} - \frac{b}{2} - \frac{1}{2}\right)\left(\frac{b^2}{2} + \frac{b}{3} - \frac{1}{2}\right)$
18. $\left(\frac{x^2}{5} - \frac{2x}{3} + \frac{1}{5}\right)\left(\frac{x^2}{3} - \frac{x}{9} + \frac{1}{3}\right)$
19. $\left(a^2 - \frac{2a}{3} + \frac{1}{4}\right)\left(2a^2 - \frac{1}{3}a - 1\right)$
20. $\left(\frac{y^2}{4} + 2y - 3\right)\left(y^2 - \frac{y}{2} + \frac{1}{3}\right)$

19. Division of Monomials. Since, by the definitions,

$$a^m = a \cdot a \cdot a \cdots (a \text{ written } m \text{ times})$$

and $$a^n = a \cdot a \cdot a \cdots (a \text{ written } n \text{ times}),$$

we see that $$\frac{a^m}{a^n} = a \cdot a \cdot a \cdots (a \text{ written } m - n \text{ times}),$$

or $$\boldsymbol{a^m \div a^n = a^{m-n}}$$

The exponent of the quotient of two powers of a number is the difference of the exponents of the two powers. Thus:

$$a^5 \div a^2 = a^{5-2} = a^3; \ x^7 \div x^3 = x^{7-3} = x^4;$$

and $$a^{2r-s} \div a^r = a^{2r-s-r} = a^{r-s}$$

To divide a monomial, such as $30\,x^3y^5$, by another, such as $-5\,xy^2$, first divide 30 by -5, then x^3 by x, and then y^5 by y^2. Thus:

$$30\,x^3y^5 \div -5\,xy^2 = \frac{30}{-5}\,x^{3-1}y^{5-2} = -6\,x^2y^3$$

20. ORAL EXERCISES

State the quotients:

1. $x^5y^4 \div x^2y$

2. $18\,x^5y^4 \div 9\,x^2y$

3. $15\,ab^2 \div ab$

4. $-24\,y^3 \div y^2$

5. $8\,x^3y \div -2\,x^3$

6. $4.5\,b^3 \div 1.5\,b$

7. $4.5\,r^4 \div .15\,r$

8. $4.5\,a^6 \div .015\,a^2$

9. $.45\,y^5 \div .15\,y^4$

10. $.45\,s^7 \div .015\,s$

Literal Exponents

11. $x^{2a} \div x^a$ ANS. x^{2a-a}, or x^a

12. $x^{3a} \div x^a$

13. $x^{3a} \div x^2$

14. $r^{3x} \div r^2$

15. $y^{r+s} \div y^r$

16. $x^{2a}y^b \div x^ay^c$

17. $x^{5a}y^{3b} \div x^{2a}y^b$

18. $a^rb^s \div ab$

19. $x^ay^b \div x^by^a$

20. $e^x \div e^{x-1}$

21. $e^{x+2} \div e^{x-1}$

22. $e^{6x} \div e^{2x}$

23. $e^{2x} \div e^2$

21. Division of Polynomials by Monomials. To divide $6x^3y - 8x^2y^2 + 4xy^3$ by $2xy$, divide each term of the dividend by the divisor, obtaining $3x^2 - 4xy + 2y^2$.

An expression like $(9x^6 - 3x^5 - 6x^4) \div 3x^2$ means that each term in the parenthesis is to be divided by $3x^2$. The parentheses are put around the dividend to indicate that *all* the terms in the parenthesis are to be divided. If the parentheses are omitted, as in $6x^2 - 8x \div 2x$, then only the term just to the left of the division sign is divided, and the result in this case is $6x^2 - 4$.

22. ORAL EXERCISES

The pupil will acquire a useful habit if he will read the work in ex. 1, for example, as

$$20a - 15b = 5(4a - 3b);$$

that is, "$20a - 15b$ equals 5 times the quantity $4a - 3b$."

Find the quotients:

1. $(20a - 15b) \div 5$
2. $(6y^2 - 4y) \div 2y$
3. $(15r^2 - 10r) \div 5r$
4. $(18a^3 + 15a^2) \div 3a$
5. $(15a^3 + 21a^2) \div 3a^2$
6. $(24y^3 + 12y^2) \div 6y^2$
7. $(20b^3 - 15b) \div 5b$
8. $(16x^5 + 8x^3) \div -4x^3$
9. $(12x^5 - 36x^2) \div -4x^2$
10. $(18x^4 - 27x) \div -9x$
11. $(12x^3 + 4x^2 + 4x) \div 4x$ Notice here that the quotient is $3x^2 + x + 1$. We cannot omit the last term, $+1$, as we do when writing just x for $1x$.
12. $(20y^3 + 5y^2 - 5y) \div 5y$
13. $(30x^4 - 24x^3 + 12x^2 - 6x + 6) \div 6$
14. $(30a^5 - 36a^4 - 15a^3 - 3a^2) \div 3a^2$
15. $(24y^{10} - 40y^7 + 32y^3 - 4y) \div 4y$
16. $(6a^3b^3 + 8a^3b^2 - 18ab^2 - 12ab^3) \div 2ab$
17. $(-36r^4 - 27r^3 + 18r^2 - 45r) \div -9r$
18. $(10x^4y + 18x^3y^2 - 16x^2y^3 + 2xy^4) \div 2xy$
19. $(b^2 - b + 1) \div -1$
20. $(a^2 - 2ab - b^2) \div -1$
21. $(-x^2 + xy - y^2) \div -1$
22. $(-ab + bc - ac) \div -1$

23. Division of Polynomials. To divide a polynomial, such as $8x^2 - 2x - 7$, by $2x - 3$, proceed as follows:

1. Divide the first term of the dividend, $8x^2$, by the first term of the divisor, $2x$.

$$\begin{array}{r} 4x \qquad\qquad \\ 2x-3\overline{)8x^2-2x-7} \end{array}$$

2. Multiply the divisor, $2x - 3$, by $4x$, getting $8x^2 - 12x$.

$$\begin{array}{r} 4x \qquad\qquad \\ 2x-3\overline{)8x^2-\ \ 2x-7} \\ 8x^2-12x \quad\ \ \end{array}$$

3. Subtract the product from the terms above it, getting $+10x$.

$$\begin{array}{r} 4x \qquad\qquad \\ 2x-3\overline{)8x^2-\ \ 2x-7} \\ \underline{8x^2-12x} \quad\ \ \\ +10x \quad\ \ \end{array}$$

4. Bring down the next term of the dividend, -7, and then start again with steps 1, 2, 3. The next term of the quotient is $+5$ because $10x \div 2x = +5$.

The finished work is shown at the right.

$$\begin{array}{r} 4x \qquad\ \ +5 \\ 2x-3\overline{)8x^2-\ \ 2x-\ \ 7} \\ \underline{8x^2-12x} \qquad\ \ \\ +10x-\ \ 7 \\ \underline{10x-15} \\ +\ \ 8 \end{array}$$

Hence $\dfrac{8x^2 - 2x - 7}{2x - 3} = 4x + 5 + \dfrac{8}{2x - 3}$

It is a good habit *to write the result* of the work in the above form and not merely to do the dividing. Notice also that here you cannot write $4x + 5 \dfrac{8}{2x - 3}$, but must insert a sign between the quotient and the fraction.

To check the work, multiply the quotient by the divisor and then add the remainder. The result should be the dividend. Or you may check as follows:

$$\frac{8x^2 - 2x - 7}{2x - 3} = 4x + 5 + \frac{8}{2x - 3}$$

Substitute $x = 4$: $$\frac{8(16) - 8 - 7}{8 - 3} = 16 + 5 + \frac{8}{8 - 3}$$

$$\tfrac{113}{5} = 21 + \tfrac{8}{5}$$

$$22\tfrac{3}{5} = 22\tfrac{3}{5}$$

In checking, you may not substitute a number that makes a denominator equal zero. In the above work you could not use $x = 1\frac{1}{2}$. The reason for this is given on page 10.

24. EXERCISES

Divide:

1. $12x^2 + 7x - 4$ by $4x + 5$
2. $15x^2 - 11x - 6$ by $3x - 1$
3. $4x^2 - 4ax - 15a^2$ by $2x - 5a$
4. $10x - 8 + x^3 - 5x^2$ by $x - 2$ Before dividing rearrange the terms in the dividend: $x^3 - 5x^2 + 10x - 8$.
5. $6x^3 + 13x - 12 - 11x^2$ by $3x - 4$
6. $12y^3 - 9y + 4y^2 + 4$ by $3y + 1$
7. $15x^3 - 13x^2y + 11xy^2 - 6y^3$ by $3x - 2y$
8. $4a^3 - 7a^2b + 7ab^2 - 3b^3$ by $4a - 3b$
9. $x^3 - 21x + 20$ by $x - 4$ Since the dividend does not contain an x^2 term, write it as $x^3 + 0x^2 - 21x + 20$, or leave a blank space for the terms in x^2. This is not absolutely essential, but is very convenient, as will be seen when the division is performed.
10. $4x^3 + x - 12$ by $2x - 3$
11. $3x^3 + 8x^2 - 8$ by $x + 2$
12. $x^3 + 2x^2 - 11x - 10$ by $x^2 - 2x - 3$
13. $3x^3 - 4x^2y + 7xy^2 - 2y^3$ by $x^2 - xy + 2y^2$

In ex. 14 to 17 find only the first four terms of the quotients:

14. $a^8 - b^8$ by $a - b$
15. $a^8 + b^8$ by $a + b$
16. $a^7 - b^7$ by $a - b$
17. $a^7 + b^7$ by $a + b$
18. $\frac{a^3}{3} + \frac{5a^2}{12} - \frac{a}{12} + \frac{3}{2}$ by $\frac{a}{3} + \frac{3}{4}$ Here we may avoid fractions by first multiplying both dividend and divisor by 12. Why does this not change the quotient?
19. $b^3 - \frac{b^2}{6} + \frac{b}{6} - \frac{1}{3}$ by $\frac{b}{2} - \frac{1}{3}$
20. $\frac{c^3}{3} + \frac{c^2}{4} - \frac{c}{3} - \frac{1}{4}$ by $\frac{c}{3} + \frac{1}{4}$

25. Parentheses. Whenever a collection of terms is to be treated as a single quantity, we place parentheses, ( ), around that quantity. Thus, on page 17 we used parentheses to show that all the terms of $2x^2 - 3xy + 4y^2$ were to be multiplied by $5x$, and on page 21 we used parentheses to group all the terms of the dividend, $9x^6 - 3x^5 - 6x^4$.

Similarly, if we wish to subtract $2x - 3$ from $5x + 4$ we place parentheses around the subtrahend, $2x - 3$, and write: $5x + 4 - (2x - 3)$. We may then subtract in either of two ways:

1. We may change the sign of the subtrahend and then add:

$$5x + 4 - (2x - 3) = 5x + 4 + (-2x + 3)$$
$$= 5x + 4 - 2x + 3.$$

2. We may multiply the subtrahend by -1 and then add:

$$5x + 4 - (2x - 3) = 5x + 4 - 2x + 3$$

The first method seems more natural, but the second method enables us to work more rapidly.

EXAMPLES

1. $5(2a - 3b) = 5 \cdot 2a - 5 \cdot 3b = 10a - 15b$

2. $-6(x^2 - 3x + 2) = -6x^2 + 18x - 12$

3. $4(6x + 5) - 7 = 24x + 20 - 7 = 24x + 13$

Notice in ex. 3 that the term -7 is not inside the parenthesis and hence is not multiplied by 4.

4. $3 + 4(2x - 5) = 3 + 8x - 20 = 8x - 17$

Here we do not add 3 to $+4$, but add 3 to the result of multiplying $2x - 5$ by 4.

5. $8 - 3(2x - 1) = 8 - 6x + 3 = -6x + 11$

Here also we do not add 8 to -3, but add 8 to the result of multiplying $2x - 1$ by -3.

6. $4(3x - 5) - 3(2x + 1) = 12x - 20 - 6x - 3$
$$= 6x - 23$$

In this case we multiply the first quantity, $3x - 5$, by 4 and multiply the second quantity, $2x + 1$, by -3. We then add the similar terms.

26. EXERCISES

Simplify the following expressions as much as possible:

1. $3(2x + 4) - 5$
2. $4(5x - 1) - 2$
3. $-4(x - 3) + 2$
4. $6(-2x + 5) - 3$
5. $8(5 - 4x) - 7$
6. $5(2 + a) - 4a$
7. $3 + 4(5x - 7)$
8. $8 - 6(2x + 3)$
9. $5x - 4(x + 1)$
10. $8 - 3(2x + 4)$
11. $7 - 4(-x + 2)$
12. $3x - 5(x + 4)$
13. $2 - 5a - (3a - 7)$
14. $4 - 3b - (2 - 3b)$
15. $2 + 4y - (6 - 4y)$
16. $7 - 3b - (4b - 6)$
17. $10 - (2a - 3) + 5$
18. $3b - (a - b) + b$
19. $2x - y - (x - y)$
20. $4r - (r + s - t)$
21. $x - (y + z) - z$
22. $x - (y - z) - y$
23. $a - (2a - 5b) + b$
24. $a - (a + b - c)$
25. $5(3x - 2) + 4(3x + 1)$
26. $4(5x - y) - 3(x + 2y)$
27. $3(4a + 1) + 2(5a - 3)$
28. $-4(r + 1) - 3(5 - 3r)$
29. $2(5 - 3y) - 6(y + 4)$
30. $6(3 + 5b) - 3(6 - 4b)$
31. $7(1 - s) + 2(4 - 2s)$
32. $3(x - y) - 2(x + y)$
33. $5(2a - 3b - 4c) + 3(5a + 3b - 2c)$
34. $6(\frac{1}{3}a - \frac{1}{6}b + \frac{1}{2}c) + 12(\frac{1}{4}a + \frac{1}{2}b - \frac{1}{6}c)$
35. $30(\frac{2}{5}x - \frac{1}{3}y - \frac{3}{10}z) - 24(\frac{1}{8}x - \frac{5}{12}y + \frac{3}{4}z)$
36. $20(\frac{1}{5}x - \frac{3}{10}y + \frac{1}{4}z) + 16(\frac{5}{8}x - \frac{3}{4}y + \frac{1}{4}z)$
37. Does $a - b + c$ equal $a - (b + c)$ or $a - (b - c)$?

It is sometimes necessary (see page 46) to place certain terms of a quantity in a parenthesis preceded by a minus sign. What signs are changed when this is done?

In ex. 38 to 43 insert the last three terms of each quantity in a parenthesis preceded by a minus sign:

38. $a^2 - x^2 + 2xy - y^2$
39. $a^2 - x^2 - 2xy - y^2$
40. $a^2 - 1 - 2x - x^2$
41. $a^2 - 9 + 6x - x^2$
42. $9a^2 - r^2 + 2rs - s^2$
43. $x^2 - y^2 - 2yz - z^2$

CHAPTER II

LINEAR EQUATIONS. PROBLEMS. FORMULAS

27. Equations and Identities. In previous work we have used the equal sign = in two ways:

1. To connect algebraic expressions which are equal for some particular value or values of the letters. Thus, we write $3x = x + 8$, which is true only if $x = 4$; and we write $y = x + 6$, which is true only if one of the numbers, y, is 6 larger than the other number x.

2. To connect two algebraic expressions which are equal for every value of the unknown letters. Thus, we write:

$$x^2 - y^2 = (x + y)(x - y)$$

and

$$x^2 - 5x + 6 = (x - 3)(x - 2)$$

These equations are true regardless of the values of x and y.

In future work we shall aim to be more exact in our language and hence we shall say that:

1. An equation that is true only for certain values of the letters in it is called a *conditional equation* or, briefly, an *equation.*

The particular numbers for which an equation is true are called *roots* of the equation, and are said to *satisfy* the equation.

2. An equation that is true for all values of the letters in it is called an *identity.*

We also call an equation like $5(9 - 2) = 45 - 10$ an identity because the two members will contain exactly the same terms if the indicated operations are performed.

28. Axioms Used in Solving an Equation. To find the roots of an equation, we use the following axioms:

1. The same number may be added to both members of an equation.
2. The same number may be subtracted from both members of an equation.
3. Both members of an equation may be multiplied by the same number.
4. Both members of an equation may be divided by the same number except zero. (See page 10.)

The first axiom may also be stated as a rule:

If any number is added to one member of an equation, the same number must also be added to the other member.

Restate the other axioms in the same way.

In actual work we do not consciously use the first two axioms as stated above, but abbreviate them by a process called *transposition*.

If we add $2x$ to both members of $5x = 35 - 2x$, the work appears thus:

Before adding: $5x = 35 - 2x$
After adding: $2x + 5x = 35$

The term $-2x$ has disappeared from the right member and the term $+2x$ has appeared in the left member.

If we subtract 6, or add -6, to both members of the equation $3x - 6 = 9$, the work appears thus:

Before adding: $3x - 6 = 9$
After adding: $3x = 9 + 6$

The term -6 has disappeared from the left member and the term $+6$ has appeared in the right member.

PRINCIPLE. *Any term may be omitted from one member of an equation provided the opposite term (the term with its sign changed from + to − or from − to +) is added to (or written in) the other member.*

EXAMPLES OF THE SOLUTIONS OF EQUATIONS

EXAMPLE 1. Solve $5x-(2x-9)=35-4(x+3)$.

Perform the operations indicated by the parentheses:

$$5x-2x+9=35-4x-12$$

Transpose: $5x-2x+4x=-9+35-12$

Simplify: $7x=14$

Divide by the coefficient of x: $x=2$

Check the solution:

$$\begin{aligned} 5x-(2x-9)&=35-4(x+3)\\ 10-(4-9)&=35-4(2+3)\\ 10-(-5)&=35-4(5)\\ 15&=15 \end{aligned}$$

EXAMPLE 2. Solve $\dfrac{x+9}{5}-4=\dfrac{x+1}{6}-\dfrac{5x-19}{10}$.

First change this equation into a new one in which there are no fractions by multiplying each term by 30, the L. C. M. of the denominators. It is advisable to write the multiplier in front of each term and to place parentheses around the binomial numerators.

$$\mathbf{30}\frac{(x+9)}{5}-\mathbf{30}\cdot 4=\mathbf{30}\frac{(x+1)}{6}-\mathbf{30}\frac{(5x-19)}{10}$$

Divide the multiplier, 30, by each denominator.

$$\mathbf{6}(x+9)-120=\mathbf{5}(x+1)-\mathbf{3}(5x-19)$$

Solve this equation as in example 1. $16x=128$, or $x=8$.

Check the work by substituting $x=8$ in the given equation:

$$\begin{aligned} \frac{x+9}{5}-4&=\frac{x+1}{6}-\frac{5x-19}{10}\\ \frac{17}{5}-4&=\frac{9}{6}-\frac{21}{10}\\ \frac{102}{30}-\frac{120}{30}&=\frac{45}{30}-\frac{63}{30}\\ -\frac{18}{30}&=-\frac{18}{30} \end{aligned}$$

Notice in the check that you did not try to get rid of the fractions (for then any error in the solution might be repeated in the check) but you changed all fractions to the same denominator. Also notice that you substituted $x=8$ in the given equation, not in some other form

29. EXERCISES

Solve and check the following equations. Do ex. 1 to 15 orally.

1. $3 + x = 0$
2. $3 + x = 1$
3. $4 - x = 0$
4. $5 - x = 1$
5. $0 = 5 - x$
6. $3y = 2$
7. $5y = 6$
8. $4 = 3y$
9. $6 = 7y$
10. $5y = 0$
11. $-a = 2$
12. $2a = -5$
13. $-a = -7$
14. $1 = -a$
15. $-9a = 2$
16. $8(x - 5) = 5(x - 2)$
17. $5(y - 6) + 7y = 6$
18. $4r = 1 - 2(r - 1)$
19. $3(s - 4) = 2s - 14$
20. $3t - 4(t + 1) = 2$
21. $9 + 4(3y - 7) = 29$
22. $14 - 9a = 4(4 - 3a)$
23. $-b + 3(b + 6) = 21$
24. $3y = 18 - 2(y - 1)$
25. $2z = 5 + 3(4z + 7)$
26. $8 = 3y + 2(y - 1)$
27. $9 = 5r - 3(r + 1)$
28. $4b - 3 - (2b - 1) = 7$
29. $5m - 2 = m - (2m + 1)$
30. $3 - (4n - 5) = 6n - 2$
31. $5t - 7 = 1 - (8 + t)$
32. $-6y = 1 - (3 + 4y)$
33. $8 - (r + 7) = r + 7$
34. $6a - (2 - a) = 12$
35. $5b - (9 - 2b) = -9$
36. $\dfrac{x - 18}{4} + \dfrac{2x + 40}{5} = \dfrac{x + 10}{2}$
37. $\dfrac{y - 5}{6} + \dfrac{y - 7}{3} = \dfrac{y - 11}{9}$
38. $\dfrac{1 - z}{2} - \dfrac{1 + 2z}{3} = 5 - \dfrac{1 + z}{4}$
39. $\dfrac{a + 10}{2} + \dfrac{7 - a}{3} = \dfrac{a + 6}{4} + \dfrac{13}{2}$
40. $\dfrac{u}{6} - \dfrac{3u - 40}{5} + 7 = \dfrac{u + 30}{2}$
41. $\dfrac{3(v - 4)}{4} + 6 = \dfrac{5v - 12}{6} - \dfrac{2(v + 3)}{3}$

Solve and check the following equations:

42. $\frac{2(x-18)}{3}+1=\frac{x}{7}$

43. $\frac{9y+1}{5}-\frac{7}{15}=y$

44. $z-\frac{3z+4}{6}=\frac{1}{2}$

45. $\frac{23+3y}{6}+y=\frac{5}{6}$

46. $t-\frac{3t-2}{5}=\frac{2}{3}$

47. $\frac{1-2a}{3}+\frac{4a}{5}=\frac{4}{15}$

48. $\frac{5-4b}{4}=\frac{1}{4}-\frac{b}{3}$

49. $\frac{3c+4}{5}-\frac{5}{8}=\frac{3c}{10}$

50. $\frac{6r+7}{3}-\frac{3r+2}{2}=\frac{5}{2}$

51. $\frac{1-2s}{5}+\frac{39}{30}=\frac{5-4s}{6}$

52. $\frac{9r+4}{6}-\frac{3r-2}{10}=\frac{7}{15}$

53. $\frac{6u-5}{3}-\frac{4-3u}{4}=1$

54. $\frac{7-6v}{5}-\frac{1-3v}{10}=\frac{5}{2}$

55. $\frac{4s+1}{3}-\frac{2s+1}{5}=\frac{3}{5}$

56. $\frac{3x-4}{3}-\frac{3(x-1)}{4}=\frac{5}{3}$

57. $\frac{5y+1}{4}-\frac{4y+6}{9}=2$

58. $10(\frac{1}{2}t-3)-2t=6-6(\frac{1}{3}t-4)$

59. $12(\frac{1}{2}+\frac{1}{3}r)-18(\frac{5}{6}-\frac{1}{3}r)=15(\frac{1}{5}r+\frac{1}{3})$

60. $8(\frac{3}{4}s-\frac{1}{2})+10(\frac{3}{5}-\frac{1}{2}s)=2-16(\frac{3}{8}-\frac{1}{4}s)$

61. $\frac{1}{2}(12x-10)=\frac{1}{3}(9x-12)$

62. $\frac{1}{4}(20y-12)=\frac{1}{5}(15y-10)$

63. $\frac{2}{3}(3a+6)-\frac{3}{4}(4a-8)=30-5a$

64. $\frac{3}{2}(b+2)-\frac{3}{5}(18-b)=11-\frac{1}{2}(b-4)$

65. $\frac{1}{3}(m+2)-\frac{1}{5}(m+5)+\frac{1}{4}(m+4)=16$

66. $14+5(2y+.4)=12(y+.5)-10(.2-y)$

67. $10(2r-1.5)-15(r-.3)=.25-(r+1.75)$

68. $2s-(.3s+.2)=6(.4s-.7)-(s-4.6)$

69. $2(6-.1t)-(.3t+.4)=3t-5(.56-.5t)$

70. $.35x-3.427=9.862-(.654-.315x)$

72. Solve the equation $2x + a^2 = 4(a - 2) + 18$ for x after putting $a = 5$.

73. If $s = vt + \frac{1}{2}gt^2$, find the value of s when $v = 9$, $t = 6$, and $g = 32.2$.

74. In the equation $2ax = a^2 - 2b^2 + 5c^2$ put $a = 6$, $b = 5$, $c = -2$ and then solve for x.

75. In the equation $a^2 = 2bx - ab - cx$ put $a = 5$, $b = -3$, $c = 4$ and then solve the equation for x.

76. In the equation $dL(W - P) = Wh$ substitute $L = 4\frac{1}{2}$, $P = 6$, $h = 4$, $d = 1\frac{1}{2}$ and then solve for W.

77. In the equation $n(tT - C) = C$ put $T = 56{,}000$, $n = \frac{4}{3}$, $t = 3$ and then solve for C.

Substitute the given numbers in the following equations and then find the value of the unknown number:

78.	$F = \dfrac{W}{r + 1}$	$r = 5$, $W = 30$
79.	$E = C(R + n)$	$C = 8, E = 112, R = 12$
80.	$W = P\dfrac{2R}{R - r}$	$P = 140, R = 12, r = 4$
81.	$d = \dfrac{n + 2}{p}$	$n = 16$, $p = 1.5$
82.	$S = \dfrac{n}{2}(a + l)$	$n = 26$, $a = 1$, $l = 101$
83.	$l = a + (n - 1)d$	$a = 10$, $d = 5$, $n = 60$
84.	$T = 2\pi r(r + h)$	$h = 8$, $r = 2.5$, $\pi = 3.14$
85.	$T = \pi r(r + l)$	$r = 1.5, l = 14.5, \pi = 3.14$
86.	$Wr = PR$	$r = 3$, $R = 15$, $W = 200$
87.	$s = vt + \frac{1}{2}gt^2$	$s = 744$, $g = 32$, $t = 3$
88.	$c^2 = a^2 + b^2 - 2ap$	$c = 25$, $a = 20$, $b = 10$
89.	$A = P(1 + rt)$	$A = 600, P = 400, t = 10$
90.	$A = P(1 + r)^2$	$A = 441$, $r = .05$

30. Preliminary Steps in Solving Problems. When solving a problem we first determine what the unknown numbers or quantities are, then decide which one to represent by x or some other letter, and by what to represent the others.

The unknown numbers may be compared in various ways, such as by stating how much larger or smaller one is than another, or how many times larger one is, or by stating the sum of the two numbers.

31. ORAL EXERCISES

1. If n represents an unknown number, state:

(*a*) the number that is 3 times as large as n
(*b*) the number that is 5 larger than n
(*c*) the number that is 6 smaller than n
(*d*) the number such that the sum of it and n is 15
(*e*) the number that is half as large as n
(*f*) the number that exceeds n by 12
(*g*) the number that exceeds twice n by 5

2. If a man is y years old to-day, what was his age 5 yr. ago? What will be his age 3 yr. hence?

3. A boy has d dimes and 3 more nickels than dimes. How many nickels has he? What is the value of the nickels in cents? What is their value expressed in dollars?

4. One man is traveling r miles an hour; another man goes 4 mi. an hour faster. How many miles will the second man go in 5 hr.? in h hours? in m minutes?

5. One man travels m miles; another man goes 50 mi. farther. How far does the second man travel? If the latter is going at the rate of 30 mi. an hour, how many hours will he need to make the journey?

6. One man travels m miles; another man goes 60 mi. farther. If the second man needs 8 hr. for the trip, what is his rate in miles an hour?

32. Suggestions for Solving Problems. The pupil should *study this section now,* and also *refer to it* whenever he has difficulty with problems like those on pages 35 to 37.

1. *Percentages.* A man sold his house for \$5950, losing 15% of the cost. What was the cost of the house?

Let the cost $= x$ dollars. Then $x - 15\%$ of $x = 5950$. 15% may be written either as $\frac{15}{100}$ or as a decimal, .15.

2. *Consecutive Integers.* Whole numbers that differ by 1, like 9, 10, 11 or -8, -7, -6, are consecutive integers. If we represent the first integer by x, how shall we represent the others?

Consecutive *even* integers, like 2, 4, 6 or -8, -6, -4, differ by 2; that is, each integer is 2 larger than the preceding one. Consecutive *odd* integers, like 3, 5, 7 or -3, -1, $+1$, also differ by 2. In either case, if n is the first integer, how shall we write those which follow it?

By trying both even and odd integers for n, find out what kinds of integers are represented by the expressions:

(*a*) $n - 1, n, n + 1$ (*c*) $2n - 1, 2n, 2n + 1$
(*b*) $n - 2, n, n + 2$ (*d*) $2n - 2, 2n, 2n + 2$

3. *Rates.* A and B start from the same town and move in opposite directions, A going 25 mi. an hour and B going 30 mi. an hour. B starts 3 hr. later than A. How many hours after A starts will they be 240 mi. apart?

Let us say that the answer is t hours. The information in the problem may be tabulated in the diagram:

	Time	Rate	Distance
A	t	25	$25t$
B	$t - 3$	30	$30(t - 3)$

An equation can then be formed by using the fact that:
A's distance + B's distance = their distance apart.

4. *Rate of Working.* A can do certain work in 18 days and B in 12 days. After A has worked alone for 3 days, he hires B to help him finish. How many days does B work?

Let us say that B works x days. Since B does $\frac{1}{12}$ of the work each day, his part of the work is $\frac{x}{12}$. A works $(x + 3)$ days, doing $\frac{1}{18}$ of the job each day. His part of the total work is $\frac{x+3}{18}$.

An equation can then be formed by using the fact that

$$\left\{\begin{matrix}\text{Total part of work}\\ \text{done by A}\end{matrix}\right\} + \left\{\begin{matrix}\text{Total part of work}\\ \text{done by B}\end{matrix}\right\} = \left\{\begin{matrix}\text{Entire}\\ \text{work}\end{matrix}\right\}$$

5. *Mixtures and Alloys.* How many ounces of metal that is 75% gold should be mixed with some metal that is 40% gold to make 70 oz. of 55% gold?

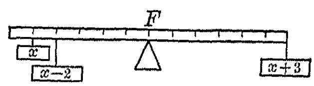
The following picture of the information is useful:

$$\boxed{\begin{matrix}x \text{ ounces}\\ 75\% \text{ gold}\end{matrix}} + \boxed{\begin{matrix}(70 - x) \text{ ounces}\\ 40\% \text{ gold}\end{matrix}} = \boxed{\begin{matrix}70 \text{ ounces}\\ 55\% \text{ gold}\end{matrix}}$$

Hence $\frac{75}{100}x + \frac{40}{100}(70 - x) = \frac{55}{100}(70)$ because

$$\left\{\begin{matrix}\text{Amount of gold}\\ \text{in first metal}\end{matrix}\right\} + \left\{\begin{matrix}\text{Amount of gold}\\ \text{in second metal}\end{matrix}\right\} = \left\{\begin{matrix}\text{Amount of gold}\\ \text{in final alloy}\end{matrix}\right\}$$

6. *Levers.* When several weights are balanced on a lever, the product of any weight by its distance from the fulcrum is called the *leverage* of the weight. If the weights x, $x - 2$, and $x + 3$, for example, are at the distances 5, 4, and 6 respectively from F, then the leverages are $5x$, $4(x - 2)$, and $6(x + 3)$.



It is proved in physics that the lever will balance if

$$\left\{\begin{matrix}\textit{the sum of the leverages at the left of the fulcrum}\\ = \textit{the sum of the leverages at the right of the fulcrum.}\end{matrix}\right.$$

33. PROBLEMS

1. When wheat is ground into flour, 18% of the weight is lost. What was the original weight of some wheat that weighed 2460 lb. after grinding?

2. A man sold his automobile for $1260, which was 40% less than the car cost. What was the cost of the car?

3. A man wishes to invest $9000, part at 7% and the remainder at 4%, so as to average 6% on his money. How much should he invest at each rate?

SUGGESTION. $\left\{\begin{matrix}\text{Income on 1st}\\ \text{investment}\end{matrix}\right\} + \left\{\begin{matrix}\text{Income on 2d}\\ \text{investment}\end{matrix}\right\} = \left\{\begin{matrix}\text{Income on}\\ \text{total fund}\end{matrix}\right\}$

4. A man wishes to invest $10,000, part at 5% and the remainder at $7\frac{1}{2}$%, so as to have an income of $700. How much should he invest at each rate?

5. How can $12,000 be divided in two parts, one part invested at $5\frac{1}{2}$% and the remainder at 7%, so as to get an average income of 6%?

Problems About Integers

6. Find three consecutive integers whose sum is 72.

7. Find three consecutive even integers whose sum is 72.

8. Are there three consecutive odd integers whose sum is 72?

9. Write any three consecutive integers and find their sum. Repeat the work, using x to represent the first integer. Is the sum exactly divisible by 3?

How would you prove a statement like: The sum of any three consecutive integers is divisible by 3?

10. Find the sum of any four consecutive integers and then prove the statement: The sum of any four consecutive integers is always 6 more than 4 times the smallest integer.

11. Discover a statement like the one in ex. 10 for the sum of five consecutive integers; of five consecutive even integers; of five consecutive odd integers.

Problems About Rates

12. A freight train leaves New York for Chicago at a rate of 28 mi. an hour. Six hours later an express train leaves New York, going 48 mi. an hour. In how many hours will the express train overtake the freight train?

13. I have 5 hr. at my disposal. How far may I ride at 9 mi. an hour so that I can return within the given time if I walk back at the rate of $3\frac{1}{2}$ mi. an hour?

14. A boatman needed 6 hr. to go upstream a certain distance and $1\frac{1}{2}$ hr. to return. If the rate of the current in the river is 2 mi. an hour, at what rate can the man row in still water? (The following suggestion may prove helpful.)

SUGGESTION. If the rate at which the man can row in still water is r, then his rate upstream is $(r-2)$ and his rate downstream is $(r+2)$.

15. Some boys needed $3\frac{1}{2}$ hr. to go upstream a certain distance and $2\frac{1}{2}$ hr. to return. If the rate of the current in the river is $\frac{3}{4}$ mi. an hour, find the rate the boys row in still water.

Problems About Rate of Working

16. One pump can fill a tank in 10 hr. and a smaller pump can do it in 12 hr. After the smaller pump has been going 3 hr., the larger pump is also started. In how many more hours will the tank be filled?

17. One newspaper press can print an edition in 56 min. and another in 35 min. After the first press has been running 8 min. it gets out of order and the other press is started. How soon can it finish the work?

18. A tank can be filled in 10 hr. by one pipe and can be emptied in 12 hr. by another pipe. How many hours will it take to fill the tank if both pipes are open?

19. One pipe can fill a tank in 10 hr.; another, in 6 hr.; a third pipe can empty the tank in 8 hr. How long will it take

Problems about Mixtures

20. How many ounces of metal that is 65% copper should be mixed with some metal that is 50% copper to make 75 oz. of metal that is 60% copper?

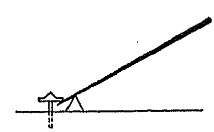
21. A grocer mixes some coffee selling at 40¢ a pound with some that sells at 70¢ a pound to make 120 lb. worth 60¢ a pound. How many pounds of each kind does he mix?

22. If a bushel of oats is worth 60¢ and a bushel of corn is worth 90¢, how many bushels of each should a miller use to make a mixture of 100 bu. worth 84¢ a bushel?

23. How many gallons of cream containing 25% butter fat should be added to 84 gal. of milk containing 3% fat to produce milk with 4% butter fat?

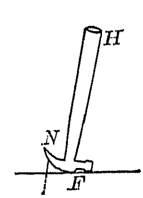
24. How much pure gold must be added to 50 oz. of gold that is 14 carats fine (that is, $\frac{14}{24}$ pure) to make an alloy that will be 22 carats fine?

Problems about Levers



25. In the figure at the right, a force of 160 lb. must be exerted at the end of a crowbar 3 ft. long to pull a spike out of the wood. If the fulcrum is 4 in. from the spike, what force is exerted on the spike?

26. How heavy a stone can a man lift by exerting a force of 100 lb. with a crowbar 6 ft. long if the fulcrum is 18 in. from the stone?



27. The arms of a lever need not form a straight line. In the figure, the length FH (from the hand to the point where the hammer rests) is the right arm of a lever, and the length FN is the left arm of the lever. If $FH = 10$ in. and $FN = 1\frac{1}{2}$ in., what force must be exerted to

34. Formulas. In previous work certain rules have been written in algebraic language. Thus:

Area of a rectangle	$A = lw$
Area of a triangle	$A = \frac{1}{2}bh$
Area of a circle	$A = \pi r^2$
Circumference of a circle . . .	$C = 2\pi r$

A *formula* is a rule of computation written in algebraic language.

To say that $A = \pi r^2$ is a formula for the area of a circle *in terms of the radius* means that we must know the radius to find the area. The phrase *in terms of the radius* also emphasizes the fact that the area depends on the radius; that is, if the radius changes then the area also changes.

Formulas are used extensively in all sciences and trades. The workman in a shop or office must be able to *use* formulas even though he may not understand how they are found.

Whenever you use a formula:

1. State the formula.
2. State the numerical values of the various letters.
3. State which is the unknown number.
4. Show the numerical work. (Do not write this work on "scratch" paper and then throw it away.)

EXAMPLE. Find the area of a trapezoid 9 in. high whose upper base is 5.8 in. and lower base 14.6 in.

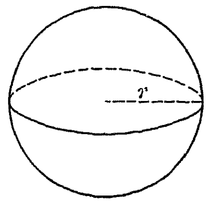
The work for this problem should be written thus:

Formula: $A = \frac{h}{2}(a+b)$	5.8
	14.6
Given: $h = 9$, $a = 5.8$, $b = 14.6$	20.4
Find A. $A = \frac{9}{2}(5.8 + 14.6)$	9
	183.6

35. **EXERCISES — FORMULAS**

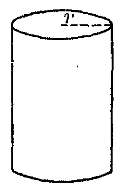
Use $\pi = 3.14$. Since only three figures of π are used, only the first three figures of the result will be accurate.

1. The volume of a sphere is $\frac{4}{3}$ times π times the cube of its radius. Write this rule as a formula. Use the formula to find the volume if the radius is 2 in.



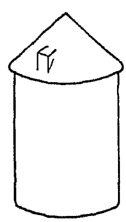
2. The surface area of a sphere is 4 times the area of the largest circle that can be cut from it. Write this rule as a formula. Find the surface area of a sphere whose radius is 3 in. Next, find the area if the radius is 6 in., and then answer the question: If the radius is doubled, is the area doubled?

3. A right circular cylinder has a circle for its base. The volume is equal to the area of the base, πr^2, multiplied by the height, h. Hence $V = \pi r^2 h$. Find the volume when $r = 4$ in. and $h = 9$ in.



Find the volume of a cylinder 1 ft. high, the radius of the base being 2.8 in.

4. A farmer built a cylindrical silo, the base being a circle 6 ft. in radius. Find the amount of material in the silo when it is filled to a height of 8 ft.; 16 ft.; 24 ft.



5. Two different silos have the same height, 30 ft., but the base of one is 12 ft. in radius and the base of the other is 6 ft. in radius. Is one volume double the other? Compare the volumes.

6. If, in ex. 4, a certain amount of silage fills the silo to a depth of 8 ft., will twice the amount fill the silo to a depth of 16 ft.?

Can you explain why, in ex. 4, doubling the depth doubles the volume while, in ex. 5, doubling the radius will more than double the volume?

36. EXERCISES

In the following exercises represent each quantity by some letter, as c, d, n, and write the formula that states how one of the quantities depends on the other.

1. If sugar costs 8¢ a pound, the cost, c, depends on the number of pounds, n, which are bought.

2. William has \$20 now and saves \$4 a week. How many dollars, d, will he have after n weeks?

3. Henry has \$12 now and saves \$5 a week. How much money will he have after n weeks?

4. Use the equations found in ex. 2 and 3 to solve the problem: In how many weeks will Henry and William have the same amount of money?

5. A tank contains 300 gal. of water; 12 gal. flow out each minute. What is the number of gallons, g, remaining in the tank after m minutes?

6. A tank contains 60 gal. of water; 18 gal. are poured into it each minute. What is the number of gallons in the tank after m minutes?

7. A steamer leaves New York with 800 tons of coal. If it burns 15 tons an hour, what is the number of tons, T, left after h hours?

If you have difficulty with the above exercises, use the following method of studying:

In ex. 7, for example, first try a few definite numbers. After 1 hr. the steamer has 785 tons left; after 2 hr. it has 770 tons, etc. Hence to find the amount remaining, multiply the number of hours by 15 and then subtract from 800.

In each problem notice which numbers are multiplied, added, subtracted, etc.

8. A society charges \$60 for an initiation fee and \$8 a year for dues. Write the equation stating the total amount, A, which has been paid by a member after y years.

9. A man is 120 mi. from Georgetown and is moving away from it at the rate of 25 mi. an hour. What is his distance from the town after h hours?

10. A man is 250 mi. from Morley and is moving toward it at the rate of 12 mi. an hour. How far is he from the city after some number of hours?

11. A man's salary is \$2500 a year and increases \$120 annually. What is his salary after n years?

12. The rate on a telegram to a certain town is 32¢ for the first 10 words and 3¢ for each additional word. What is the cost, c, for w words? Assume that w is more than 10.

Test the correctness of your equation by using it to find the value of c when $w = 15$. You should find $c = 47$.

13. If the rate for a long-distance telephone call is 65¢ for the first 3 minutes and 5¢ for each additional minute, what is the cost for m minutes?

Test your equation to see if $c = 95$ when $m = 9$.

14. A telephone company charges \$2.70 a month plus \$.06 for each call over 30 calls. If n is the number of calls and is more than 30, what is the charge?

Test your equation to see whether $c = 3.30$ when $n = 40$.

15. The parcel post charges for a certain zone are 8¢ for the first pound and 2¢ for each additional pound. What is the relation between the postage and the weight?

Test your equation to see if $p = 12$ when $w = 3$.

16. The parcel post charges for a certain zone are 9¢ for the first pound and 4¢ for each additional pound. What is the relation between the postage and the weight?

Test your equation to see if $p = 37$ when $w = 8$.

17. A gas company charges its customers \$1.25 a thousand for the first 10,000 cu. ft. of gas, and \$.80 a thousand for all gas over 10,000 cu. ft. What is the charge for m cubic feet of gas, assuming that m is more than 10,000?

CHAPTER III

SPECIAL PRODUCTS. FACTORING

37. Products of Binomials. $(ax + b)(cx + d)$. The pupil has already learned how to find the products of binomials mentally or "by inspection" as follows:

EXAMPLE. $(2x - 3)(5x + 4) = ?$

The first term is found by multiplying the first terms in each parenthesis.	$(2x \quad)(5x \quad) = 10x^2$
The middle term is found by multiplying the two end terms, multiplying the two nearest terms, and adding these results.	$(2x \quad)(\quad + 4) = 8x$ $(\quad - 3)(5x \quad) = \underline{-15x}$ $\qquad\qquad\qquad\quad -7x$
The last term is found by multiplying the last terms in each parenthesis.	$(\quad - 3)(\quad + 4) = -12$

Hence $(2x - 3)(5x + 4) = 10x^2 - 7x - 12$

38. Squares of Binomials. Since $(3x - 5)^2$ means $(3x - 5)(3x - 5)$, we could find the product as in the above example, but we can obtain the result more quickly by noticing that:

1. The first term of the product is $9x^2$. It is the square of the first term, $3x$, of the binomial.

2. The second or middle term is $-30x$. It is *twice* the product of the two terms, $3x$ and -5, of the binomial.

3. The third term of the product is $+25$. It is the square of the last term, -5, of the binomial.

39. EXERCISES

State the indicated products and powers:

1. $(3x + 2)(4x + 5)$
2. $(3x + 2)(4x - 5)$
3. $(3x - 2)(4x - 5)$
4. $(3x - 2)(4x + 5)$
5. $(2a - 1)(3a + 2)$
6. $(2a^2 - 1)(3a^2 + 2)$
7. $(a + b^2)(a + 3b^2)$
8. $(a^2 + b)(a^2 + 3b)$
9. $(ab + 3)(ab - 5)$
10. $(a + 3b)(a - 5b)$
11. $(3a - b)(a - 3b)$
12. $(3x - 1)(3x - 2)$
13. $(3x - y)(3x - 2y)$
14. $(3x^2 - y)(3x^2 - 2y)$
15. $(5 - 6t)(2 + 3t)$
16. $(5 - 6t^3)(2 + 3t^3)$
17. $(a^2 + 3)^2$
18. $(a^2 + 3b)^2$
19. $(ab - 3)^2$
20. $(ab + 3c)^2$
21. $(a - 3b^2)^2$
22. $(7ab - 4)^2$
23. $(2x + 5y)^2$
24. $(2rs - 5)^2$
25. $(2x - y)^2$
26. $(x + 2y)^2$
27. $(x + \frac{1}{4})^2$
28. $(6r - \frac{1}{3})^2$
29. $(9x + \frac{1}{2})^2$
30. $(\frac{1}{2}a + 3)^2$
31. $(\frac{1}{3}y - 5)^2$

32. $(3x - 2)(3x + 2)$ The product is $9x^2 - 4$. The middle term of the product has disappeared because it is $+6x - 6x$, which equals *zero*.

33. $(4a + 5)(4a - 5)$
34. $(4a + 5b)(4a - 5b)$
35. $(a + 2b)(a - 2b)$
36. $(a^2 - 2b)(a^2 + 2b)$
37. $(ab - 2)(ab + 2)$
38. $(3 - 5s)(3 + 5s)$
39. $(5r + 1)(5r - 1)$
40. $(5x - y)(5x + y)$
41. $(4r - 5s^2)(4r + 5s^2)$
42. $(a^5 + b^5)(a^5 - b^5)$

43. $(x^a - 3)(x^a + 8)$ Ans. $x^{2a} + 5x^a - 24$

44. $(y^n + 4)(y^n - 6)$
45. $(x^{3a} + 2)(x^{3a} - 5)$
46. $(a^{2x} + 4)(a^{2x} - 3)$
47. $(e^x - 5)(e^x + 2)$
48. $(e^{3x} + 1)(e^{3x} - 4)$
49. $(e^{2x} - 3)(e^{2x} + 5)$
50. $(e^x + 1)^2$
51. $(e^x - 1)^2$
52. $(e^{2x} + 3)^2$
53. $(e^x + a)^2$
54. $(3e^x + 5)^2$
55. $(4e^x - 3)^2$

40. Factoring $ax^2 + bx + c$. To factor any quantity such as $21x^2 + 2x - 8$, we must find two binomials whose product is this quantity. The process consists chiefly in guessing various terms and then trying them to see if they are correct. Write first

$$21x^2 + 2x - 8 = (\ — \ — \ )(\ — \ — \ )$$

We are to find the four missing terms in the parentheses. Very likely the first terms are $3x$ and $7x$ because their product is the first term, $21x^2$. Hence we write

$$21x^2 + 2x - 8 = (3x \quad — \quad)(7x \quad — \quad)$$

The two missing terms must be numbers whose product is -8. Such numbers are -8 and $+1$, or $+8$ and -1, or -2 and $+4$, or $+2$ and -4. Hence

$$21x^2 + 2x - 8 = \begin{cases} (3x - 8)(7x + 1) & \text{or} \\ (3x + 8)(7x - 1) & \text{or} \\ (3x - 2)(7x + 4) & \text{or} \\ (3x + 2)(7x - 4) & \end{cases}$$

or some other variation.

We multiply the binomials shown, and thereby find out which combination is correct. It may happen that none is correct because the factors of $21x^2$ may be $21x$ and $1x$.

41. Factoring Trinomial Squares. When we use the above method to factor $9x^2 - 30xy + 25y^2$, we find that the two factors are the same, and hence we write

$$9x^2 - 30xy + 25y^2 = (3x - 5y)^2$$

We should be able to recognize such cases very quickly. On page 42, § 38, we saw that a trinomial is the square of some binomial if two of the terms of the trinomial are squares, and the remaining term is twice the product of the square roots of the terms that are squares. Thus:

$$9a^2 - 42ab + 49b^2 = (3a - 7b)^2$$

and

$$x^2 + \tfrac{2}{3}x + \tfrac{1}{9} = (x + \tfrac{1}{3})^2$$

42. EXERCISES

1. Examine the exercises below and state which of the quantities are squares of a binomial.

Find, by experiment, the factors of the following:

2. $21x^2 + x - 10 \quad = (7x + \quad)(3x - \quad)$
3. $8x^2 + 2x - 15 \quad = (4x - \quad)(2x + \quad)$
4. $6x^2 - 19x + 10 \quad = (2x - \quad)(3x - \quad)$
5. $2x^2 - 5x + 3 \quad = (2x - \quad)(x - \quad)$
6. $12x^2 + 25x - 22 = (4x + \quad)(3x - \quad)$

7. $12z^2 + 11 \cdot z - 15$
8. $12r^2 - 8r - 15$
9. $15b^2 + 31b - 24$
10. $15b^2 + 2b - 24$
11. $15b^2 - 46b + 24$
12. $r^2 - 9r + 18$
13. $r^6 - 9r^3 + 18$
14. $r^2s^2 - 9rs + 18$
15. $r^4 + 3r^2 - 18$
16. $x^2 + 10x + 25$
17. $x^2 - 10x + 25$
18. $8a + a^2 + 16$
19. $a^2 + 8ab + 16b^2$
20. $4c^2 + 9 - 12c$
21. $x^2y^2 - 16xy + 64$
22. $a^{10} + 2a^5 + 1$
23. $4 + 36x + 81x^2$
24. $6 - 5y + y^2$
25. $5 + 6y + y^2$
26. $49 - 42x + 9x^2$
27. $m^2 - 8m + 16$
28. $m^2 - 10m + 16$
29. $2r + 1 + r^2$
30. $x^2 + x + \frac{1}{4}$
31. $3s^2 - 2s - 5$
32. $3s^8 - 2s^4 - 5$
33. $6x^4 + 5x^2 - 4$
34. $20a^2 + 9ab - 20b^2$
35. $x^6 - 20x^3 + 100$
36. $t^2 - 44t - 45$
37. $45t^2 + 46t + 1$
38. $6t^2 - 31t + 5$

39. $3x^{8a} - 14x^{4a} - 5$ ANS. $(3x^{4a} + 1)(x^{4a} - 5)$

40. $e^{2x} + 10e^x + 24$
41. $e^{6x} + 7e^{3x} + 12$
42. $e^{2x} + 8e^x + 16$
43. $x^{4a} - 10x^{2a} + 25$
44. $y^{2b} - y^b - 6$
45. $2a^{6x} + 7a^{3x} + 6$
46. $9b^{2y} + 3b^y - 2$
47. $3c^{2m} - 10c^m + 8$
48. $e^{2x} + 2e^x + 1$
49. $6y^{2b} - 17y^b + 12$

43. The Difference of Two Squares. A quantity like $4x^2 - 9y^2$ is called the *difference of two squares.* It can be factored thus: $4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$.

One factor is the sum of the square roots, $2x$ and $3y$; *the other factor is the difference of the square roots.*

Many other quantities can be written as the difference of two squares, as the following illustrations show:

1. $$a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2$$
$$= (a + b + c)(a + b - c)$$

2. $a^2 - b^2 - 2bc - c^2 = a^2 - (b^2 + 2bc + c^2)$ Here the last three terms are grouped within a parenthesis. Since we wish to have the parenthesis preceded by a minus sign (in order to have the *difference* of two quantities) we change the signs of $-b^2$, $-2bc$, and $-c^2$, and these terms appear inside the parenthesis as b^2, $+2bc$, and $+c^2$.

One factor is $a + (b + c)$; the other is $a - (b + c)$. To avoid having a parenthesis within a parenthesis, as in the expression $(a + (b + c))(a - (b + c))$, we then remove the inner parentheses in the usual way. Hence

$$a^2 - b^2 - 2bc - c^2 = a^2 - (b + c)^2$$
$$= (a + b + c)(a - b - c)$$

3. $x^4 + x^2 + 1$ If the second term of this quantity were $+2x^2$, then the quantity would be the square of $x^2 + 1$. We may make it $2x^2$ if we also subtract x^2.

$$x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2 = (x^2 + 1)^2 - x^2$$

which is the difference of two squares and can be factored thus: $(x^2 + 1 + x)(x^2 + 1 - x)$

4. $$x^4 - 14x^2y^2 + 25y^4 = (x^4 - 10x^2y^2 + 25y^4) - 4x^2y^2$$
$$= (x^2 - 5y^2)^2 - (2xy)^2$$
$$= (x^2 - 5y^2 + 2xy)(x^2 - 5y^2 - 2xy)$$

5. $4a^4 + 1$ Write this as: $(4a^4 + 4a^2 + 1) - 4a^2$
$= (2a^2 + 1)^2 - (2a)^2 = (2a^2 + 1 + 2a)(2a^2 + 1 - 2a)$

44. EXERCISES

Factor the following polynomials. In some exercises it is necessary to rearrange the terms to show the two squares.

In ex. 1 to 5 multiply the factors which you find, in order to check the work and test your understanding of factoring.

1. $x^2 + 6\,xy + 9\,y^2 - z^2$
2. $x^2 - (9\,y^2 + 6\,yz + z^2)$
3. $x^2 - 9\,y^2 + 6\,yz - z^2$
4. $x^2 - y^2 - 6\,y - 9$
5. $x^2 - a^2 + 4\,a - 4$
6. $4\,x^2y^2 - x^2 - 2\,xy - y^2$
7. $a^2 - 1 - 2\,x - x^2$
8. $x^2 - b^2 - 6\,xy + 9\,y^2$
9. $a^2 + 8\,ab - 16 + 16\,b^2$
10. $a^2 + 2\,a - c^2 + 1$
11. $a^2 - 2\,ab + b^2 - x^2 - 2\,xy - y^2$
12. $4\,b^2 + 4\,ab + a^2 - x^2 + 4\,xy - 4\,y^2$
13. $r^2 - c^2 + 4\,s^2 - 4\,x^2 + 4\,rs + 4\,cx$
14. $a^2 + 16\,b^2 - 8\,ab + 30\,xy - 9\,x^2 - 25\,y^2$
15. $r^2 - a^2 + s^2 - b^2 - 2\,rs - 2\,ab$
16. $m^4 + 5\,m^2 + 9$
17. $25\,y^4 - 19\,y^2 + 1$
18. $25\,y^4 + y^2 + 1$
19. $25\,y^4 - 11\,y^2 + 1$
20. $25\,y^4 + 9\,y^2 + 1$
21. $4\,a^4 + b^4$
22. $x^4 + 4$
23. $x^4 + 4\,y^4$
24. $64\,r^4 + s^4$
25. $4\,a^{4n} + 1$

Miscellaneous Exercises

26. $(x - y)^2 + 7(x - y) + 12$ If you cannot factor this quantity, replace $(x - y)$ by R, and write $R^2 + 7\,R + 12$. After factoring this expression, replace R by $(x - y)$.
27. $4(a - b)^2 - 12\,c(a - b) + 9\,c^2$
28. $x^2 - 2\,x(r + s) - 35(r^2 + 2\,rs + s^2)$
29. $a^2 - 5\,ax + 5\,ay + 6\,x^2 - 12\,xy + 6\,y^2$
30. $y^2 + 6\,ay - 6\,by + 5\,a^2 - 10\,ab + 5\,b^2$
31. $y^2 + 2\,ry + 2\,sy + r^2 + 2\,rs + s^2$

45. Prime Factors. In factoring $36x^2 + 15x - 6$ by the usual method, some pupils may get the result shown in (1) below and others may get the result shown in (2):

(1) $36x^2 + 15x - 6 = (9x + 6)(4x - 1)$

(2) $36x^2 + 15x - 6 = (3x + 2)(12x - 3)$

Notice that in arithmetic, too, we may get different results by factoring. For example, $70 = 35 \cdot 2$; also $70 = 7 \cdot 10$. We can also write $70 = 7 \cdot 5 \cdot 2$. These three numbers, 7, 5, and 2, are called the *prime factors* of 70 because 7, 5, and 2 cannot be factored without using fractions.

In order to find the prime factors of a polynomial, it is advisable to do the work in three steps:

1. Find any monomial factor. This means that you should look carefully at each term in the polynomial to see if the terms are all divisible by some monomial factor. Thus:

$$36x^2 + 15x - 6 = 3(12x^2 + 5x - 2)$$
$$2ax^2 + 8ax - 24a = 2a(x^2 + 4x - 12)$$

2. After dividing by the monomial factor and writing the result as in step 1, examine the quotient, and factor it by any of the known methods, if possible. Thus:

$$\begin{aligned} 36x^2 + 15x - 6 &= 3(12x^2 + 5x - 2) \\ &= 3(3x + 2)(4x - 1) \end{aligned}$$

3. Examine the factors obtained in step 2 to see if any of them can be factored further.

EXAMPLES

1. $\begin{aligned} m^4 - 4m^3 - 45m^2 &= m^2(m^2 - 4m - 45) && \text{by step 1} \\ &= m^2(m - 9)(m + 5) && \text{by step 2} \end{aligned}$

2. $\begin{aligned} &2x^5 - 28x^3 + 90x \\ &= 2x(x^4 - 14x^2 + 45) && \text{by step 1} \\ &= 2x(x^2 - 9)(x^2 - 5) && \text{by step 2} \\ &= 2x(x + 3)(x - 3)(x^2 - 5) && \text{by step 3} \end{aligned}$

3. $a^4 - 16 = (a^2 + 4)(a^2 - 4) = (a^2 + 4)(a + 2)(a - 2)$

46. EXERCISES

Find the prime factors of the following quantities.

For oral work the pupils may do just step 1; that is, state any monomial factor and the quotient.

√ 1.	$12x^3 - 2x^2 - 24x$	21.	$x^3 - 9x$
2.	$21x^3 - 27x^2 - 30x$	22.	$x^3 + 9x$
3.	$2a^2b - 12ab^2 + 18b^3$	23.	$x^3 - 9xy^2$
4.	$x^2 - 4x$	24.	$x^2y - 9y^3$
5.	$x^3 - 4x$	25.	$3r^2 + 3r - 36$
6.	$2x^3 - 8x$	√ 26.	$3r^4 - 3r^3 - 36r^2$
7.	$2x^3 - 8xy^2$	27.	$\frac{1}{2}a^2 - \frac{1}{2}b^2$
8.	$3y^2 - 18y + 24$	28.	$\pi R^2 - \pi r^2$
9.	$3ay^2 - 18ay + 24a$	29.	$2\pi R - 2\pi r$
10.	$a^4 - 4a^2b - 21b^2$	30.	$2\pi rh + 2\pi r^2$
11.	$a^3 - 4ab^2$	31.	$\pi r^2 + \pi rh$
12.	$6a^3 - 24ab^2$	32.	$P + Prt$
13.	$7a^4 - 28a^2b^2$	33.	$63a - 7a^3$
14.	$10ax^2 + 7ax + a$	34.	$7a^2 - 21a$
√ 15.	$6r^3 - r^2 - r$	35.	$5r^2 - 10r$
16.	$3c^4 - 3d^4$	36.	$5r - 10s$
17.	$2s^5 - 32s$	37.	$r^2 + 2rs + s^2 - t^2$
18.	$r^8 - s^8$	38.	$a^4 + a^3b - 2a^2b^2$
19.	$h^4 + h^2k - 72k^2$	39.	$r^5 + 4rs^4$
20.	$ab^4 - 13ab^2 + 36a$	40.	$a^8 - 5a^4 + 4$

41. $a^2x^2 + 2abx^2 + b^2x^2 - 9c^2$

42. $3a^2 - 6a + 3 - 3c^2$

43. $4x^3y^2 - x^3 - 2x^2y - xy^2$

44. $4r^8x + 8r^4s^4x + 9s^8x$

√ 45. $a^2r^3 - r - 2rx - rx^2$

46. $c^6 + 4c^4 + 16c^2$

47. Factoring by Grouping Terms. To factor a quantity like $4ax + 4ay + 7bx + 7by$, or any quantity containing more than three terms, we first look at the various terms to see in what ways they are alike or different. In the above quantity we notice that:

The first two terms are both divisible by $4a$.

The last two terms are both divisible by $7b$.

Hence

$$4ax + 4ay + 7bx + 7by = 4a(x + y) + 7b(x + y)$$

This expression states that $(x + y)$ is multiplied first by $4a$ and then by $7b$. In other words, $(x + y)$ is multiplied by $(4a + 7b)$; or

$$4a(x + y) + 7b(x + y) = (4a + 7b)(x + y)$$

EXAMPLE.
$$\begin{aligned} &4ax + 12ay + 5bx + 15by \\ &= 4a(x + 3y) + 5b(x + 3y) \\ &= (4a + 5b)(x + 3y) \end{aligned}$$

48. EXERCISES

1. Factor the quantity in the above example by using the idea that the first and third terms are divisible by x, while the second and fourth terms are divisible by $3y$.

Factor the following polynomials:

2. $5ax + 8bx + 5ay + 8by$
3. $4ax + 7bx + 4ay + 7by$
4. $2ax + bx + 2ay + by$
5. $2cx - bx + 8ac - 4ab$
6. $2ax - 6ay + bx - 3by$
7. $x^2 + bx + cx + bc$
8. $ax + bx + ab + x^2$
9. $ar - cr + as - cs$
10. $10ax - 5x + 2ay - y$
11. $3hx - 3kx + hy - ky$
12. $2ax - 2ay - 3bx + 3by$

SUGGESTION. If the first two terms are divided by $2a$, the quotient is $(x - y)$. The last two terms should therefore be divided by $-3b$ so that the quotient will be the same as that found from the first two terms. Thus:

$$\begin{aligned} 2ax - 2ay - 3bx + 3by &= 2a(x - y) - 3b(x - y) \\ &= (2a - 3b)(x - y) \end{aligned}$$

Factor:

13. $2\,ax - 2\,bx - 3\,ay + 3\,by$

14. $2\,ac - 6\,ad - bc + 3\,bd$

15. $ac - ad - bc + bd$

16. $ac - ad + bc - bd$

17. $ac - 2\,ad - 2\,bc + 4\,bd$

18. $rx - 2\,sx - 3\,ry + 6\,sy$

19. $rx + 2\,sx - 3\,ry - 6\,sy$

20. $5\,rx - 5\,tx - 3\,ry + 3\,ty$

21. $2\,ac - 4\,ad - 3\,bc + 6\,bd$

22. $ru - 2\,rv - su + 2\,sv$

23. $2\,a + 2\,b - ax^2 - bx^2$

24. $x^2 - xy + 3\,y - 3\,x$

25. $abx^2 - axy - bxy + y^2$

26. $abc - acd - bd + a^2c^2$

27. $a^2 - b^2 + 5\,a - 5\,b$ This can be written as $(a + b)(a - b) + 5(a - b)$, which equals $(a + b + 5)(a - b)$.

28. $x^2 - y^2 + 3\,x - 3\,y$

29. $x^2 - y^2 - 3\,x + 3\,y$

30. $a^2 - b^2 + ac + bc$

31. $a^2 - b^2 + a - b$

32. $6\,x^2 - 6\,y^2 + 5\,x - 5\,y$

33. $5\,x^2 - 5\,y^2 + 3\,x + 3\,y$

34. $a^3 - a^2b + 3\,a^2 - 3\,ab$

35. $a^4c - a^3bc + 5\,a^3c - 5\,a^2bc$

36. $(a^2 + 2\,ab + b^2) + 3(a + b)$

37. $a^2 - 2\,ab + b^2 + 5\,a - 5\,b$

38. $a^2 - 2\,ab + b^2 - 7\,a + 7\,b$

39. $ax + ay + bx + by + cx + cy$

40. $br - 2\,r + bs - 2\,s + bt - 2\,t$

Find mentally the following products:

41. $(2\,x - y)(a + b + c)$ Multiply the second quantity by $2\,x$ and then by $-\,y$.

42. $(2\,a + 3\,b)(r + s + t)$

43. $(x^2 - y^2)(2\,a - 3\,b + 4\,c)$

44. $(x + y + z)(a + b + c)$

45. $(x - a)(y - b)$

46. $(l + 3)(w - 2)$

47. $(a + b)(x - y)$

48. $(x + 3)(x^2 - 5)$

49. $(2\,r + 3)(4\,s - 7)$

50. $(2\,l + 5)(3\,w - 4)$

49. The Factor Theorem. The meaning and usefulness of this theorem can best be seen from the following study:

1. Divide $x^2 - 7x + 9$ by $x - 2$. What is the remainder? What is the value of the dividend when $x = 2$?

2. Divide $x^2 - 7x + 9$ by $x - 3$. What is the remainder? What is the value of the dividend when $x = 3$?

3. Divide $x^2 - 7x + 9$ by $x + 2$. What is the remainder? What is the value of the dividend when $x = -2$?

4. At the right, the division of $x^2 + ax + b$ by $x - n$ is shown. What is the remainder? Substitute n for x in the dividend. How does the result compare with the remainder?

$$\begin{array}{r|l}
 & x + (n + a) \\ \hline
x - n & x^2 + ax \qquad + b \\
 & \underline{x^2 - nx} \\
 & (n + a)x \qquad + b \\
 & \underline{(n + a)x - n^2 - an} \\
 & \qquad n^2 + an + b
\end{array}$$

This work shows that we can find the remainder by substituting n for x in the dividend. If we are not interested in the quotient, this is a very useful process.

The method, however, applies only to quantities that are *rational* and *integral;* that is, to quantities that do not contain any indicated square roots of x, cube roots of x, etc. and that do not contain x in a denominator.

Suppose that when we substitute n for x, we find that the value of the quantity is zero. This means that the remainder is *zero* when the quantity is divided by $(x - n)$; that is, the quantity is exactly divisible by $(x - n)$.

We have therefore found a new way of factoring:

If a rational integral expression in x *has the value* 0 *when* n *is substituted for* x, *then* $x - n$ *is one of its factors.*

Notice that n may be a positive or a negative number. If the quantity equals 0 when -3 is substituted for x, then the factor is $x - (-3)$, or $x + 3$.

EXAMPLE 1. Factor $x^3 + 2x - 3$.

When $x = 1$, then $x^3 + 2x - 3 = 1 + 2 - 3$, or 0.

Hence $x - 1$ is a factor. Dividing the given quantity by $x - 1$ you find that the quotient is $x^2 + x + 3$. Hence

$$x^3 + 2x - 3 = (x - 1)(x^2 + x + 3)$$

The quotient, in this case $x^2 + x + 3$, should always be examined carefully to see if it is factorable.

EXAMPLE 2. Factor $x^3 - x^2 - x - 15$.

You can see at a glance that when $x = 1$ or when $x = -1$ the quantity does not equal zero. It is useless to try $x = 2$ because if $x - 2$ is a factor, then the term -15 must be divisible exactly by 2, which it is not. Hence it is best to try next the numbers 3, -3, 5, -5, etc.

50. EXERCISES

1. What is the value of $x^3 - 7x^2 + 4x + 12$ when $x = 1$? when $x = 2$? when $x = 3$? when $x = -1$? when $x = -2$? State some of the factors of this quantity.

Find the prime factors of the following quantities.

Show in your work *all* the numbers that you tried, as in ex. 1, and show also the division.

2. $x^3 + 5x - 6$	**9.** $x^3 + x^2 - 10x + 8$
3. $x^3 - 8x + 8$	**10.** $y^3 - 6y^2 + 11y - 6$
4. $x^3 - 4x^2 + 9$	**11.** $r^3 + 2r^2 - 9r - 18$
5. $a^3 + a + 2$	**12.** $r^3 + 4r^2 - 17r - 60$
6. $x^3 - 4x + 3$	**13.** $x^3 + 7x^2 + 14x + 8$
7. $b^3 - 3b + 2$	**14.** $2y^3 + 11y^2 + 4y - 5$
8. $x^4 - 7x^2 - 6x$	**15.** $4x^3 - 20x^2 + 33x - 18$

16. What does the quantity $a^3 - a^2b - ac^2 + bc^2$ become when b is substituted for a? Is $a - b$ a factor? Find whether $a - c$ is a factor by substituting c for a. Similarly, find whether $b - c$ is a factor by substituting c for b.

17. Find which of the quantities $(a - b)$, $(b - c)$, and $(a - c)$ are factors of $a^2b - a^2c - ab^2 + ac^2 + b^2c - bc^2$.

51. EXERCISES — FACTORING $a^3 - b^3$ AND $a^3 + b^3$

1. What does the quantity $a^3 - b^3$ become when b is substituted for a? State one factor of $a^3 - b^3$. Find the other factor by dividing $a^3 - b^3$ by the known factor.

2. What does the quantity $a^3 + b^3$ become when $-b$ is substituted for a? State one factor of $a^3 + b^3$. Find the other factor by long division.

The quantities $a^3 + b^3$ and $a^3 - b^3$ are called, respectively, *the sum of two cubes* and *the difference of two cubes.*

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

These relations should be memorized to aid in factoring similar expressions. To factor $x^3 + 27y^3$, for example, substitute x for a and $3y$ for b in the first model above.

Find the prime factors of:

3. $x^3 + 8y^3$	7. $x^3 - 8$	11. $x^4 - x^3$
4. $x^3 - 8y^3$	8. $y^3 - 125$	12. $x^6 - y^6$
5. $8r^3 - s^3$	√ 9. $8x^3 + 125$ √	13. $r^6 + s^6$
6. $27r^3 + s^3$	√ 10. $r^3 - 216$ √	14. $y^7 - y$

52. Summary. When factoring proceed as follows:

Divide the quantity by the monomial factor if there is one. The quotient may be one of the following types:

(1) $ax^2 + bx + c$	Factored by inspection.
(2) $a^2 + 2ab + b^2$	A trinomial square.
(3) $a^2 - b^2$	The difference of two squares.
(4) $ax + ay + bx + by$	Factored by grouping.
(5) $a^3 \pm b^3$	The sum or difference of two cubes.

(6) A polynomial factorable by the Factor Theorem.

Before leaving the problem, examine each factor *again* to make sure that it cannot be factored further.

53. REVIEW — FACTORING

Find the prime factors of the following:

1. $a^2 - 9b^2 + 6a + 9$
2. $4x^3 + x^2 - 8x - 2$
3. $ax^2 - x^2 + 2ax - 2x$
4. $9r^4 + 2r^2s^2 + s^4$
5. $x^5 - x^3 - 8x^2 + 8$
6. $x^3 + 6x^2 + 3x - 10$
7. $(a-b)^2 - 4(c+d)^2$
8. $a^4b^4 - 21a^2b^2 + 36$
9. $3ab + 3a - 3b - 3$
10. $2ab - 2a + 2b - 2$
11. $x^4 + x^3 + x^2 + x$
12. $ax - bx - a + b$
13. $a^4 - 1 - 2x - x^2$
14. $a^4 - 5a^2 + 4$
15. $b^4 - 29b^2 + 100$
16. $x^3 - 4x^2 + 8$
17. $(1-a)^3 - 1$
18. $y^{10} - y$
19. $a^2 + \frac{11}{30}a + \frac{1}{30}$
20. $r^2 - \frac{1}{12}r - \frac{1}{12}$
21. $3ab(a+b) + a^3 + b^3$

SUGGESTION. First factor only $a^3 + b^3$ so that the given quantity is written as $3ab(a+b) + (a+b)(a^2 - ab + b^2)$. Notice the *repetition* of the factor $(a+b)$. This factor may be temporarily replaced by a single letter, as R.

Then $3abR + R(a^2 - ab + b^2) = R(3ab + a^2 - ab + b^2)$
$= R(a^2 + 2ab + b^2) = R(a+b)^2$

Finally replace R by its value: $=(a+b)(a+b)^2 = (a+b)^3$

22. $x^3 - y^3 + 3x^2 - 3y^2$
23. $x^3 + y^3 - 3x^2 + 3y^2$
24. $x^3 + y^3 - x^2 + y^2$
25. $x^3 - y^3 - x^2 + y^2$
26. $a^4 - b^4 - a^2 + b^2$
27. $a^2 - a - b^2 - b$
28. $x(x+1)(x+2) + x^2 + 3x + 2$
29. $2x^2 + 4xy + 2y^2 - 3b(x+y) - 5b^2$
30. $a + b + 2ab + a^2 + b^2$
31. $(1+y)^2 + y(1+y)$
32. $(2x+b)(x-2b) - 7b^2$
33. $x^6 - 2x^4y^2 + x^2y^4 - y^2$
34. $a^2 + b^2 + 4 + 2ab - 4a - 4b$
35. $x^2 - a^2 + y^2 - b^2 + 2xy - 2ab$

EQUATIONS SOLVED BY FACTORING

54. Problem. A man wishes to double the size of his garden, which is 40 ft. wide and 60 ft. long, by adding the same amount to the width and to the length. How many feet should he add?

If the number of feet added is x, the new area will be $(x + 40)(x + 60)$ square feet. Hence the equation is

$$(x + 40)(x + 60) = 4800, \text{ or } x^2 + 100\,x + 2400 = 4800$$

The equation will be solved later (page 57, ex. 17). This equation differs from those studied in Chapter II in that it contains the term x^2. Such an equation is called a *quadratic* equation or an equation of the *second degree*. An equation containing x^3 is called a *cubic* equation or an equation of the *third degree*.

By the *degree of a term* we mean the sum of the exponents in that term. Thus, the term $4\,x^3y^2$ is of the fifth degree. In an equation, the term with the highest degree decides the *degree of the equation*. In advanced work it is proved that the number of roots of an equation is the same as the degree.

55. EXERCISES

Find by substitution which of the numbers at the right of equations 1 to 4 are solutions of the equation:

1. $x^2 - 7\,x = -12$ $\quad 2, 3, 4$
3. $x^2 + 2 = 3\,x$ $\quad 3, 2, 1$
2. $x^3 - 2\,x = 4$ $\quad 1, 2, 3$
4. $3\,x^2 + 2\,x = 1$ $\quad -1, 1, \frac{1}{3}$

5. What is the value of $3 \cdot 0$? of $-4 \cdot 0$? of $0 \cdot 5$?

6. Think of two numbers whose product (not sum) is zero. What must one of the numbers be? Can a product be zero unless one of the numbers is zero?

7. If $a(c + d) = 0$, we know that either $a = 0$ or else $(c + d) = 0$. Make a similar statement for the product $(a + b)(c + d) = 0$; for $x(x + 1)(x - 5) = 0$.

8. If $x = 3$ or -2, show that $(x - 3)(x + 2) = 0$.

56. The solution of an equation by factoring uses the following principle:

PRINCIPLE. *If the product of two or more factors is zero, then at least one of the factors must be zero.*

EXAMPLE. Solve the equation $2x^3 + 5x^2 = 3x$

1. Transpose all terms to the left so that one member is zero. $2x^3 + 5x^2 - 3x = 0$
2. Factor the left member. $x(2x - 1)(x + 3) = 0$
3. According to the principle either $x = 0$ or $2x - 1 = 0$ or $x + 3 = 0$
4. Hence $x = 0$ or $x = \frac{1}{2}$ or $x = -3$

Prove that all three values of x are roots of the equation.

57. EXERCISES

1. (*a*) Solve $(x - 3)(x - 2) = 6(x - 3)$ by performing the multiplications, transposing, factoring, etc. What are the two roots of the equation?

(*b*) If the equation in (*a*) is divided by $(x - 3)$, we get $x - 2 = 6$. What is the root of the new equation? What root of the given equation was lost?

(*c*) Show that the root lost in (*b*) can be recovered by setting the divisor equal to 0, and solving this equation.

By factoring, solve for x or y:

2. $x^3 - x^2 - 6x = 0$
3. $x^2 - x = 0$
4. $2x^2 - 6x = 0$
5. $x^2 - 25 = 0$
6. $2y^2 + 5y = 3$
7. $2y^3 - 5y^2 = 3y$
8. $y^3 - 3y^2 - 4y = 0$
9. $8y^2 - 10y = 3$
10. $9y^2 - 12y + 4 = 0$
11. $x^3 + 2x^2 - x = 2$
12. $y^3 - 3y^2 + 12 = 4y$
13. $y^3 - 6y^2 + 11y = 6$
14. $x^3 - 7x - 6 = 0$
15. $2x^3 + x^2 + 6 = 13x$
16. $3x^4 + 9x^2 - 2x = 10x^3$
17. $x^2 + 100x = 2400$

58. Prime Factors. In this chapter we have assumed that factors involving radicals are not to be considered. Thus, we regard $x^2 - 5$ as a prime quantity although

$$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$$

Also $x^4 - 19\,x^2 + 25$ is written as $(x^2 - 5)^2 - 9\,x^2$ rather than as $(x^2 + 5)^2 - 29\,x^2$ because the first form leads to $(x^2 - 5 + 3\,x)(x^2 - 5 - 3\,x)$ while the second form leads to $(x^2 + 5 + x\sqrt{29})(x^2 + 5 - x\sqrt{29})$.

Likewise we say that $x^2 - 10\,x - 5$ is not factorable although it can be written as $(x - 5 + \sqrt{30})(x - 5 - \sqrt{30})$, as we shall see later. If we allow x to appear under a radical, then even such a simple quantity as $x - y$ can be factored because $x - y = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$.

Hence the decision whether we should call a quantity factorable or prime depends upon what kind of expression we are willing to consider in the factors. In general, only rational factors are required.

59. The Usefulness of Factoring. Besides aiding in the solution of equations, factoring is useful because (1) it enables us to simplify many arithmetic operations, and (2) it enables us to see quickly by what a quantity is divisible.

For example, when $a^4 - a^3b - 2\,a^2b^2$ is factored and written as $a^2(a + b)(a - 2\,b)$, we can see at once that it is divisible by a^2, by $(a + b)$, and by $(a - 2\,b)$. Further, we can also see what the quotients are because

The quotient of any product by one of its factors consists of the remaining factors. Thus:

$$\frac{(x-2)(x+3)(x-4)^2}{(x-4)^2} = (x-2)(x+3)$$

and

$$\frac{(x-2)(x+3)(x-4)^2}{(x+3)(x-4)} = (x-2)(x-4)$$

This principle is much used in the next chapter.

60. Factoring is also an essential step in finding the least common multiple (L. C. M.) of several quantities.

To find the L. C. M. of $3x^2 - 12$ and $5x^2 - 20x + 20$, we factor each quantity:

$$3x^2 - 12 = 3(x+2)(x-2)$$
$$5x^2 - 20x + 20 = 5(x-2)^2$$

The L. C. M. is $15(x+2)(x-2)^2$ because this quantity is divisible by each of the given quantities, and no other quantity of lower degree is divisible by them.

The L. C. M. of several rational integral expressions is the quantity of lowest degree, with the least numerical coefficient, that is exactly divisible by each of the given quantities.

To find the L. C. M., find the prime factors of each quantity. The L. C. M. is the product of all the different prime factors, using each factor the greatest number of times it occurs in any one quantity.

EXAMPLES

The least common multiple of

1. $6(x-3)(x+4)$ and $9(x+4)$ is $18(x+4)(x-3)$
2. $(3y+2)(y-1)$ and $y(y-1)$ is $y(3y+2)(y-1)$
3. $2(x+1)^2(x-6)$ and $x(x-6)^3$ is $2x(x+1)^2(x-6)^3$

61. EXERCISES

After factoring each quantity, find the L. C. M. of:

1. $x^2 - 2x - 3$ and $5x - 15$
2. $x^2 + 2x - 8$ and $x^2 + 4x$
3. $x^2 - 6xy + 9y^2$ and $4x - 12y$
4. $x^2 - 8x + 16$ and $3x^2 - 12x$
5. $2x^2 - x - 1$ and $2x^2 + 3x + 1$
6. $6x^2 - 6y^2$ and $4x^2 + 4xy$
7. $x^2 - xy$ and xy
8. $x - xy$ and $x + xy$
9. $x^2 + xy$ and $2x - 2y$
10. $x^2 - y^2$ and $x^2 + xy$
11. $xy + y^2$ and $x^2 + xy$
12. $x^3 - y^3$ and $5x - 5y$

(Supplementary Topics, Pages 60, 61)

62. The Square of a Polynomial. If, in the usual way, we multiply $(a + b + c + d)$ by itself, we find:

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2\,ab + 2\,ac + 2\,ad + 2\,bc + 2\,bd + 2\,cd$$

The terms in the right member may be written as

$$a^2 + b^2 + c^2 + d^2 + 2\,a(b + c + d) + 2\,b(c + d) + 2\,c(d).$$

When written in this form we see that the square of any polynomial is made up of the following terms:

1. *The square of each term in the polynomial.*

Notice the first four terms in the expansion above.

2. *Twice the product of each term by each subsequent term.*

Notice, above, that $2\,a$ multiplies $b + c + d$; that $2\,b$ multiplies $c + d$; and that $2\,c$ multiplies d.

EXAMPLE. $(x + 3\,y - 5\,z - \frac{1}{2}\,w)^2$

$$= (\;x\;)^2 + (\;3\,y\;)^2 + (\;-5\,z\;)^2 + (\;-\tfrac{1}{2}\,w\;)^2 + 2\,x(3\,y - 5\,z - \tfrac{1}{2}\,w) + 6\,y(-5\,z - \tfrac{1}{2}\,w) - 10\,z(-\tfrac{1}{2}\,w)$$

$$= x^2 + 9\,y^2 + 25\,z^2 + \tfrac{1}{4}\,w^2 + 6\,xy - 10\,xz - xw - 30\,yz - 3\,yw + 5\,wz$$

63. ORAL EXERCISES

State the squares of the following polynomials:

1. $a + b + c - d$
2. $a - b + c + d$
3. $a + b - c - d$
4. $a - b + c - d$
5. $a - b - c - d$
6. $3\,x - 5\,y + 4\,z$
7. $2\,x + y - 3\,z$
8. $x - 3\,y + z - 5\,w$
9. $a^3 + b - c^2 + d$
10. $a^3 + a^2 + a + 1$
11. $x + \frac{1}{2}\,y - \frac{1}{3}\,z + \frac{1}{4}\,w$
12. $r - \frac{1}{2}\,s + \frac{1}{3}\,t + \frac{1}{5}\,u$
13. $\frac{1}{2}\,u + v - \frac{1}{2}\,w + \frac{1}{3}\,z$
14. $\frac{1}{2}\,a - \frac{1}{3}\,b + \frac{1}{4}\,c - \frac{1}{5}\,d$
15. $x + .1\,y - .2\,z + .3\,w$

64. Square Roots of Polynomials Found by Trial. If a quantity like $9x^2 + 25y^2 + z^2 + 30xy - 6xz - 10yz$ is the square of some other quantity, we should be able to find the square root by applying the rules on page 60.

The term $9x^2$ is the square of either $+3x$ or of $-3x$.

The term $25y^2$ is the square of either $+5y$ or of $-5y$.

The term z^2 is the square of either $+z$ or of $-z$.

The other terms are twice the products of these roots. Hence the square root of the quantity is $3x + 5y + z$, or $3x - 5y - z$, or some other combination of these roots.

We can find out which is correct by examining the signs in the given quantity. The term $-6xz$ suggests $2 \cdot 3x \cdot -z$; the term $-10yz$ suggests $2 \cdot 5y \cdot -z$; and the term $+30xy$ suggests $2 \cdot 3x \cdot 5y$. The correct square root is $3x + 5y - z$.

Every quantity has two square roots one of which is the negative of the other. The square roots of the quantity above should therefore be written as: $\pm(3x + 5y - z)$.

65. EXERCISES

Find by trial the square roots of the polynomials:

1. $a^2 + 9b^2 + c^2 + 6ab - 2ac - 6bc$
2. $a^2 + 9b^2 + c^2 - 6ab + 2ac - 6bc$
3. $x^2 + y^2 + z^2 + w^2 - 2xy + 2xz - 2xw - 2yz + 2yw - 2zw$
4. $\frac{1}{4}x^2 + y^2 + \frac{1}{9}z^2 + w^2 - xy + \frac{1}{3}xz - xw - \frac{2}{3}yz + 2yw - \frac{2}{3}zw$
5. $4r^2 + 9s^2 + t^2 + 16w^2 + 12rs - 4rt - 16rw - 6st - 24sw + 8tw$

CHAPTER IV

FRACTIONS

66. Changes in the form of a fraction are based on the following principle:

PRINCIPLE. *A fraction is not changed in value if both numerator and denominator are multiplied or are divided by the same finite quantity.*

This principle is used to reduce fractions to lower terms and to change several fractions to a common denominator so that they may be combined.

67. Reduction of Fractions. A fraction can be reduced or simplified whenever its numerator and denominator can be divided by the same quantity. In order to find out by what to divide we may need to factor the numerator and the denominator. (See § 59, page 58.)

EXAMPLE 1. Reduce $\frac{8\,a^3b^2}{6\,ab^3}$.

Here the numerator and the denominator are monomials; therefore we can select the divisor, $2\,ab^2$, merely by looking at the exponents. Thus:

$$\frac{8\,a^3b^2}{6\,ab^3} = \frac{8\,a^3b^2 \div 2\,ab^2}{6\,ab^3 \div 2\,ab^2} = \frac{4\,a^2}{3\,b}$$

EXAMPLE 2. Reduce $\frac{2\,x^2 + 3\,xy - 2\,y^2}{2\,x^2 - xy}$.

$$\frac{2\,x^2 + 3\,xy - 2\,y^2}{2\,x^2 - xy} = \frac{(x + 2\,y)(2\,x - y)}{x(2\,x - y)} = \frac{x + 2\,y}{x}$$

The numerator and the denominator were divided by $(2\,x - y)$.

The x's in the result cannot be canceled because the numerator and the denominator are not *both* divisible by x.

68. Checking. To check the work, substitute numbers for the letters in the given fraction and in the reduced form of the fraction, to see whether they agree in value. Do not, however, use numbers that make a denominator *zero*, as zero may not be used as a divisor (see page 10).

EXAMPLE. To check the second example on page 62,

$$\frac{2x^2 + 3xy - 2y^2}{2x^2 - xy} = \frac{x + 2y}{x}$$

substitute $x = 3$, $y = 2$:

$$\frac{2(9) + 3(6) - 2(4)}{2(9) - 6} = \frac{3 + 4}{3}$$

$$\frac{18 + 18 - 8}{12} = \frac{7}{3}$$

The work is very likely correct since $\frac{28}{12}$ does equal $\frac{7}{3}$.

69. EXERCISES

1. Is the quantity $(x - y)(x + y) + y^2$ divisible exactly by $(x - y)$? by $(x + y)$? If this quantity were the numerator of a fraction, could the numerator be divided exactly by $(x - y)$? by $(x + y)$?

2. Answer the same questions as in ex. 1 for the quantity $y^2 - (x - y)(x + y)$; for $y^2(x - y)(x + y)$.

3. It is easy to see that the *denominator* of the fraction $\frac{(x - 9)(x + 3) + 11}{(x - 5)(x + 2)}$ is divisible exactly by $(x + 2)$; what must be done to the *numerator* before we can see whether it is divisible exactly by $(x + 2)$?

Reduce the following fractions, and check:

4. $\frac{6x^2 - x - 2}{6x^2 + 3x}$

5. $\frac{x^2 + 6x}{x^2 + 4x - 12}$

6. $\frac{x^2 + 4xy + 3y^2}{x^2 - 2xy - 3y^2}$

7. $\frac{x^2 - 6xy + 9y^2}{x^2 - 9y^2}$

8. $\frac{y^2 - 2y - 24}{y^2 + 2y - 48}$

9. $\frac{a^2 + 6a - 27}{a^2 - 12a + 27}$

70. Changes of Sign in a Polynomial. By changing the sign of a polynomial we mean changing the signs of each of the terms in it. The new expression is of course not equal to the original but is the negative of it. Thus:

$$3x^2 - 6x - 8 = -(-3x^2 + 6x + 8)$$

If the polynomial is factored, we change its sign by changing the signs in any *one* factor. Thus:

$$(2 - x)(x - 3) = -(x - 2)(x - 3)$$

as we can see by performing the multiplications.

If the signs in *two* factors are changed, we are really multiplying by -1 twice, and so the polynomial is not changed in value at all. Thus:

$$(b - a)(c - a)(x - y) = (a - b)(a - c)(x - y)$$

71. Changes of Sign in a Fraction. In a fraction three signs must be considered — the sign of the numerator, the sign of the denominator, and the sign before the fraction.

Since a fraction is an indicated quotient, we see that:

$$+\frac{+a}{+b} = +\frac{-a}{-b}; \quad +\frac{+a}{+b} = -\frac{-a}{+b}; \quad +\frac{+a}{+b} = -\frac{+a}{-b}$$

The various forms of this one fraction illustrate the following principle:

PRINCIPLE. *A fraction is not changed in value:*

If the signs of the numerator and denominator are changed.

If the sign of the numerator and the sign before the fraction are changed.

If the sign of the denominator and the sign before the fraction are changed.

The two fractions $-\dfrac{x-2}{3}$ and $-\dfrac{(x-2)}{3}$ are equal because both mean $-\frac{1}{3}(x - 2)$; that is, no change of sign is made when a parenthesis is placed around a numerator.

When signs are changed, it is best to rearrange the terms so that the first term, if possible, will be positive.

72. EXERCISES

1. Explain why $\dfrac{6\,a^2 - 6\,b^2}{4\,b - 4\,a} = -\dfrac{6\,a^2 - 6\,b^2}{4\,a - 4\,b} = -\dfrac{3(a+b)}{2}.$

What changes of sign should be made in the following fractions before anything else is done?

2. $\dfrac{6\,b^2 - 6\,a^2}{4\,a^2 + 4\,ab}$

3. $-\dfrac{18\,x - 12\,y}{6\,y - 9\,x}$

4. $\dfrac{2 - b - b^2}{3\,b^2 - 3\,b}$

5. $\dfrac{(a+b)(x-y)}{c(y-x)}$

6. $\dfrac{(x-y)(a+b)(r-s)}{(y-x)(b+a)}$

7. $\dfrac{(x-y)(x-3)(a-b)}{(x+y)(b-a)(3-x)}$

Reduce the following fractions as far as possible:

8. $\dfrac{20\,a^2 - 50\,a}{15\,a - 6\,a^2}$

9. $\dfrac{2\,a - 2\,b}{2\,b^2 - ab - a^2}$

10. $\dfrac{a - b}{2\,b^2 - ab - a^2}$

11. $\dfrac{x - 3}{x^2 - 6\,x + 9}$

12. $\dfrac{x^2 - 2\,x}{x^2 + 2\,x}$

13. $\dfrac{1 + 3\,a + 2\,a^2}{1 + a - 2\,a^2}$

14. $\dfrac{x^2 - x^3}{x^2 - 1}$

15. $\dfrac{a^2 - b^2}{a^2 - ab}$

16. $\dfrac{a + b}{a + b + (a+b)^2}$

17. $\dfrac{a^3 - 16\,ac^2}{a^2 - 8\,ac + 16\,c^2}$

18. $\dfrac{x^3 - x^2y - 3\,xy^2 + 3\,y^3}{x^2y - x^3 + 5\,xy^2 - 5\,y^3}$

19. $\dfrac{x^4 + x^2y^2 + y^4}{x^3 + y^3}$

20. $\dfrac{(x+y)^2 - 1}{ax^2 + axy + ax}$

21. $\dfrac{ab^3 - a}{a^2b^4 + a^2b^2 + a^2}$

22. $\dfrac{r^5 - r^4 + r^3 - r^2}{r^4 - 1}$

23. $\dfrac{a^3 + a^2b - ab^2 - b^3}{a^2 + 2\,ab + b^2}$

24. $\dfrac{ax - ay - cx + cy}{ax + ay - cx - cy}$

25. $\dfrac{a^3 - a^2b + ab^2 - b^3}{b^2 - a^2}$

26. $\dfrac{x^3 - 3\,x^2 + 2}{x^3 - 5\,x + 4}$

27. $\dfrac{y^3 - 7\,y + 6}{y^3 - 4\,y^2 + y + 6}$

73. Multiplication of Fractions. To multiply fractions, we form a new fraction by multiplying the numerators and multiplying the denominators. Thus:

$$\frac{5}{3} \times \frac{4}{7} = \frac{5 \cdot 4}{3 \cdot 7} = \frac{20}{21} \qquad \frac{3\,a}{b^2} \times \frac{c^2}{5\,d} = \frac{3\,ac^2}{5\,b^2d}$$

Frequently, after multiplying the numerators and the denominators, the new fraction can be simplified. Thus:

$$\frac{10\,xy^2}{7\,z} \times \frac{21\,z^3}{15x^2y} = \frac{10 \cdot 21 \cdot xy^2z^3}{7 \cdot 15 \cdot x^2yz} = \frac{2\,yz^2}{x}$$

To multiply $\frac{12\,x^2 + 11\,x - 5}{6\,x^2 - 4\,x}$ by $\frac{3\,x - 2}{4\,x + 5}$ we could multiply the numerators and then multiply the denominators of the fractions, but the result would be so complicated that it would be difficult to simplify. Hence we factor each numerator and denominator and use cancellation.

$$\frac{12\,x^2 + 11\,x - 5}{6\,x^2 - 4\,x} \cdot \frac{3\,x - 2}{4\,x + 5} = \frac{(3\,x - 1)\cancel{(4\,x + 5)} \cdot \cancel{(3\,x - 2)}}{2\,x\cancel{(3\,x - 2)} \cdot \cancel{(4\,x + 5)}}$$

$$= \frac{3\,x - 1}{2\,x}$$

We can cancel the factor $(4\,x + 5)$ in the first numerator with the factor $(4\,x + 5)$ in the second denominator and $(3\,x - 2)$ in the first denominator with $(3\,x - 2)$ in the second numerator because we are multiplying the two fractions.

74. Division of Fractions. To divide one fraction by another, invert the divisor and use it as a multiplier. Thus:

$$\frac{3}{5} \div \frac{4}{7} = \frac{3}{5} \times \frac{7}{4} = \frac{21}{20} \qquad \frac{r}{s} \div \frac{x}{y} = \frac{r}{s} \times \frac{y}{x} = \frac{ry}{sx}$$

$$\frac{2\,xy^2}{a^3b} \div \frac{x^2y}{ab^2} = \frac{2\,xy^2}{a^3b} \cdot \frac{ab^2}{x^2y} = \frac{2\,by}{a^2x}$$

$$\frac{x^2 - 6\,x + 5}{x^2 + 3\,x + 2} \div \frac{x^2 - 5\,x}{x^3 + 2\,x^2} = \frac{(x - 5)(x - 1)}{(x + 1)(x + 2)} \cdot \frac{x^2(x + 2)}{x(x - 5)} = ?$$

75. EXERCISES

Find and simplify the indicated products and quotients:

1. $\frac{2a}{9b}\cdot\frac{6ab}{5y}\cdot\frac{15by}{18x}$

2. $\frac{12a^3}{15b^5}\div\frac{2a}{3b}$

In the answers, the denominators (but not the numerators) may be left in the factored form because denominators are *never* added to other denominators, while numerators are frequently combined.

3. $\frac{x^2+2x}{x^2-3x}\cdot\frac{x^2-4x+3}{x^2+x-2}$

4. $\frac{r^4-s^4}{r^2+rs}\cdot\frac{rs+s^2}{r^2-s^2}$

5. $\frac{x}{xy-y^2}\cdot\frac{(x-y)^2}{x^2-y^2}$

6. $\frac{b}{a+b}\cdot\frac{a^2+ab}{b^2+ab}$

7. $\frac{x+1}{x}\cdot\frac{x^2+5x}{x^2+6x+5}$

8. $\frac{a^2+2ab}{a^2+2b^2}\cdot\frac{a^3+2ab^2}{a^2-4b^2}$

9. $\frac{x^2-xy}{x^3}\div\frac{x^2+xy-2y^2}{xy+y^2}$

10. $\frac{2a^2+2ab}{3a+6b}\div\frac{a^2}{a^2+2ab}$

11. $\frac{a^4-b^4}{a^2-b^2}\div\frac{a^2+b^2}{3a+3b}$

12. $\frac{a^3-b^3}{5a}\div\frac{a^2+ab+b^2}{a^2+a}$

13. $\frac{r^4+1}{r+1}\div\frac{r^4+r^2+1}{r^2-1}$

14. $\frac{x^3+8}{x^3-8}\div\frac{x+2}{x^2+2x+4}$

15. $\frac{2x^2+x-3}{5x^2-5}\div(x^2-x)$ Here multiply by $\frac{1}{x^2-x}$.

16. $\frac{x^2+xy}{x-y}\div(x+y)$

17. $\frac{x^2-4y^2}{x+3y}\div(xy-2y^2)$

18. $(a-b)\div\frac{a^2-ab}{a+2b}$

19. $(x-4)\div\frac{x^2-4x}{x+2}$

20. $\frac{a-c}{x+y}\cdot\frac{x^2+xy-2y^2}{ax-cx-2ay+2cy}$

21. $\frac{2a^3+3a^2b-2ab^2}{a^4+4a^3b+4a^2b^2}\div\frac{6a-3b}{a^4+2a^3b}$

22. $\frac{x+y+z}{x-y+z}\cdot\frac{2x+2y-2z}{3x-3y-3z}\div\frac{x^2+2xy+y^2-z^2}{x^2-2xy+y^2-z^2}$

76. Addition of Fractions. To add fractions we first change them so that they will have a common denominator.

EXAMPLES

1. Add $\frac{5}{6a^2} + \frac{7}{4a}$.

The lowest common denominator is $12a^2$ because $12a^2$ is the smallest quantity divisible by $6a^2$ and by $4a$.

Then $$\frac{5}{6a^2} = \frac{?}{12a^2} \quad \text{and} \quad \frac{7}{4a} = \frac{?}{12a^2}$$

The sum of the fractions will be a fraction with $12a^2$ for its denominator and with the sum of the numerators for its numerator.

$$\frac{5}{6a^2} + \frac{7}{4a} = \frac{10}{12a^2} + \frac{21a}{12a^2} = \frac{10 + 21a}{12a^2}$$

2. Add $\frac{x-3}{x^2+xy} - \frac{2}{xy}$.

Study the following work until you can tell what was done in each step:

$$\begin{aligned}\frac{x-3}{x(x+y)} - \frac{2}{xy} &= \frac{y(x-3)}{xy(x+y)} - \frac{2(x+y)}{xy(x+y)}\\ &= \frac{y(x-3) - 2(x+y)}{xy(x+y)}\\ &= \frac{xy - 3y - 2x - 2y}{xy(x+y)} = \frac{xy - 5y - 2x}{xy(x+y)}\end{aligned}$$

3. Add $\frac{x+1}{2x^2-6x} + \frac{4}{5x-15}$.

After factoring the denominators you see that $10x(x-3)$ is divisible by each denominator. Then

$$\frac{x+1}{2x(x-3)} = \frac{5(x+1)}{10x(x-3)}$$

$$\frac{4}{5(x-3)} = \frac{4(2x)}{10x(x-3)}$$

Hence $$\begin{aligned}\frac{x+1}{2x(x-3)} + \frac{4}{5(x-3)} &= \frac{5(x+1) + 4(2x)}{10x(x-3)}\\ &= \frac{13x+5}{10x(x-3)}\end{aligned}$$

77. **EXERCISES**

Combine into one fraction:

1. $\frac{b^2}{4a^2} - \frac{c}{a}$
2. $\frac{2}{3x^2} + \frac{4}{5xy}$
3. $\frac{5}{2ab} + \frac{3}{a^2}$
4. $\frac{5}{6a} - \frac{3}{4ab}$
5. $\frac{3}{4y} - \frac{5}{6y^2}$
6. $\frac{3}{2x} + \frac{5}{6x}$
7. $\frac{x}{y} + \frac{x}{y^2}$
8. $\frac{3r}{4s} - \frac{5}{6s}$
9. $\frac{2}{ab} - \frac{3}{b^2}$
10. $\frac{3}{4y} + \frac{5}{6y^3}$
11. $\frac{x}{y} + \frac{x}{y^3}$
12. $\frac{a}{b} - \frac{a^2}{b^3}$
13. $\frac{x}{y} + \frac{y}{x^2}$
14. $\frac{b}{a^3} + \frac{c}{a^2}$
15. $\frac{b}{a^2} - \frac{c}{ab}$
16. $\frac{3a}{a^2 + ab} + \frac{5}{a}$
17. $\frac{3b}{a^2 + ab} - \frac{5}{b}$
18. $\frac{a - b}{a^2 + ab} + \frac{5}{ab}$
19. $\frac{4}{r^2 - s^2} - \frac{3}{rs}$
20. $\frac{2}{y^2 - 1} + \frac{5}{y}$
21. $\frac{4}{9a - 9b} + \frac{5}{6a - 6b}$
22. $\frac{3}{4x - 4y} - \frac{5}{6x + 6y}$
23. $\frac{3}{8r^2 - 8r} - \frac{5}{6r - 6}$
24. $\frac{4}{9s^2 - 9s} + \frac{s + 1}{s - 1}$
25. $\frac{a + 1}{a^3 - 1} + \frac{3}{2a - 2}$
26. $\frac{a}{2b - 1} + \frac{b}{2b + 1} - \frac{a - b}{1 - 4b^2}$
27. $\frac{1}{x^3 + 8} + \frac{1}{x^3 - 8} + \frac{1}{x^2 - 4}$
28. $\frac{1}{x + y} - \frac{1}{x - y} + \frac{1}{x} - \frac{2}{y}$
29. $\frac{2}{x^2 - 3x + 2} + \frac{2}{x^2 - 5x + 6} - \frac{3}{x^2 - 4x + 3}$

78. Complex Fractions. A fraction whose numerator or denominator contains other fractions may be simplified by first multiplying its numerator and denominator by some convenient multiplier and then proceeding as usual.

EXAMPLE.
$$\frac{x-1-\frac{2}{x}}{1+\frac{2}{x}-\frac{8}{x^2}} = \frac{x^2\left(x-1-\frac{2}{x}\right)}{x^2\left(1+\frac{2}{x}-\frac{8}{x^2}\right)} = \frac{x^3-x^2-2x}{x^2+2x-8}$$
$$= \frac{x(x-2)(x+1)}{(x-2)(x+4)} = \frac{x(x+1)}{x+4}$$

Here the numerator and the denominator of the complex fraction were multiplied by x^2. The multiplier is the L. C. M. of which denominators? In ex. 5 below, the multiplier is $xy(x+y)$. What is the multiplier in ex. 11?

79. EXERCISES

Simplify:

1. $\dfrac{1+\frac{1}{a}}{a-\frac{1}{a}}$

2. $\dfrac{\frac{a}{b}+a}{\frac{b}{a^2}-\frac{a^2}{b}}$

3. $\dfrac{\frac{x}{y}-y}{\frac{y}{x^2}-x}$

4. $\dfrac{\frac{r}{t}-\frac{t}{r}}{1+\frac{2}{s}}$

5. $\dfrac{\frac{x}{y}-\frac{x}{x+y}}{\frac{y}{x}-\frac{y}{x+y}}$

6. $\dfrac{x-5+\frac{4}{x}}{1-\frac{1}{x}}$

7. $\dfrac{a-\frac{8}{a^2}}{\frac{1}{a^2}+\frac{2}{a^3}+\frac{4}{a^4}}$

8. $\dfrac{\frac{a^2}{b^2}+1+\frac{b^2}{a^2}}{\frac{a^2+b^2}{ab}-1}$

9. $\dfrac{\frac{a+b}{b}-\frac{a+b}{a}}{\frac{1}{a}-\frac{1}{b}}$

10. $\dfrac{1-\frac{2}{x}+\frac{1}{x^2}}{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}$

11. $\dfrac{\frac{1}{r}-\frac{1}{r-s}}{\frac{1}{s}-\frac{1}{r+s}}$

12. $\dfrac{(a-b)\left(\frac{1}{b}+\frac{1}{a}\right)}{\left(\frac{1}{b}-\frac{1}{a}\right)(a+b)}$

80. REVIEW — FRACTIONS

1. Multiply: $\left(1-\frac{2}{x}-\frac{8}{x^2}\right)\left(3-\frac{5}{x+2}\right)$. Change the quantity in the first parenthesis to $\frac{x^2-2x-8}{x^2}$ and the quantity in the second parenthesis to $\frac{3x+1}{x+2}$. Then find the product.

2. Show that

$$\left(1-\frac{9}{a^2}\right)\left(2+\frac{5}{a-3}\right) \text{ equals } \left(\frac{a^2-9}{a^2}\right)\left(\frac{2a-1}{a-3}\right).$$

Multiply or divide, as indicated, and simplify:

√ 3. $\left(1-\frac{7}{x}+\frac{12}{x^2}\right)\left(3+\frac{2}{x-4}\right)$

4. $\left(1+\frac{4}{a}+\frac{4}{a^2}\right)\left(2-\frac{3}{a+2}\right)$

√ 5. $\left(3-\frac{2}{y+1}\right)\div\left(9+\frac{8}{y^2-1}\right)$

6. $\left(4-\frac{b^2}{a^2}\right)\left(1-\frac{b}{2a}\right)\left(\frac{2a}{b}+1\right)$

7. $\left(x-\frac{y^2}{x}\right)\div\left(1-\frac{y}{x}\right)$

8. $\left(\frac{a}{b}+\frac{b}{a}\right)\left(1-\frac{b^2}{a^2}\right)\left(a^3+\frac{b^4}{a}\right)$

9. $\frac{b+a}{b-a}-2\left(\frac{b}{a}-\frac{b}{a-b}\right)$

10. $\frac{1-x^2}{1+y}\cdot\frac{1-y^2}{x+x^2}\left(1+\frac{x}{1-x}\right)$

√ 11. $\frac{x}{1-x+x^2}\div\left(1-2x+\frac{x^2}{1+x}\right)$

12. $\frac{1-x^3}{1+x^2}\cdot\left(\frac{2x^6}{x^6-1}-1\right)$

13. Show that $\left(1 + \dfrac{b^2 + c^2 - a^2}{2\,bc}\right)\left(1 - \dfrac{b^2 + c^2 - a^2}{2\,bc}\right)$ equals $\dfrac{(a + b + c)(-\,a + b + c)(a - b + c)(a + b - c)}{4\,b^2c^2}$.

This is an important relation in geometry and trigonometry and leads to a very useful formula for the area of a triangle for which the three sides, a, b, and c, are given.

14. Show that the formula $p = .00044\,v^2\left(A + \dfrac{s}{10}\right)$ is the same as $p = .0088\,v^2(.05\,A + .005\,s)$.

In ex. 15 to 20, which contain several signs of operations, simplify the quantities within the parentheses first; then perform the multiplications and divisions in the order in which they appear, reading from left to right. In any actual problem, the nature of the work leading to these expressions always shows what operations are to be done first.

Simplify:

15. $\dfrac{x^2 - 4\,x + 3}{x^2 + 3\,x + 2} \div \dfrac{x^2 - 9}{x^2 - 4} \cdot \dfrac{x^2 + x}{x^2 - x}$

16. $\dfrac{a^2 - b^2}{a^2 + 2\,ab + b^2} \div \dfrac{a^2 + 2\,ab - 3\,b^2}{a - b} \div \dfrac{a^2 - ab - 2\,b^2}{a^2 + 4\,ab + 3\,b^2}$

17. $\dfrac{r^2 + rs}{rs + s^2} \div \left(\dfrac{r^2 + 3\,rs + 2\,s^2}{r^2 - s^2} \cdot \dfrac{r^2 - rs - 2\,s^2}{r^2 + 2\,rs}\right)$

18. $\dfrac{ab - b^2}{a - b} - \left(\dfrac{a^2 - 2\,ab + b^2}{a^2 - b^2} \div \dfrac{a - b}{ab + b^2}\right)$

19. $\dfrac{x^2 + y^2}{x^2 - y^2} \div \left(\dfrac{x}{x - y} - \dfrac{y}{x + y}\right)$

20. $\dfrac{\dfrac{x - y}{x + y} - 1}{\dfrac{x + y}{x - y} + 1} + \dfrac{\dfrac{x + y}{x - y} - 1}{\dfrac{x - y}{x + y} + 1}$

CHAPTER V

FRACTIONAL EQUATIONS. LITERAL EQUATIONS

81. Fractional Equations. When multiplying or adding fractions you usually get a fraction as a result. Only in some cases will the product or sum happen to be a whole number. In solving an equation containing fractions, however, the first step is to change the equation into a new equation that does not contain fractions. This is called *clearing of fractions.*

To change a fractional equation into a new equation that does not contain fractions, multiply every term of the equation by the L. C. M. *of the denominators.*

EXAMPLE. Solve $\dfrac{4}{x^2 - 2x} + \dfrac{x+1}{x} = \dfrac{5x+1}{6x-12}$.

First, factor all the denominators. Then you see that a suitable multiplier would be $6x(x-2)$ because this expression is divisible by every denominator. Write the multiplier in front of each term in the equation, and place parentheses around each binomial.

$$\mathbf{6\,x(x-2)}\frac{4}{x(x-2)} + \mathbf{6\,x(x-2)}\frac{(x+1)}{x} = \mathbf{6\,x(x-2)}\frac{(5x+1)}{6(x-2)}$$

After canceling the factors in the denominators with some of the factors in the multiplier, you get the new equation:

$$24 + 6(x-2)(x+1) = x(5x+1)$$

or

$$24 + 6x^2 - 6x - 12 = 5x^2 + x$$

This is a quadratic equation and can be solved by factoring, as shown on page 57. The two solutions are $x = 3$ and $x = 4$. Each solution should be checked in the original equation.

82. Selecting the L. C. M. When the denominators of an equation contain the unknown number, care must be used in choosing the multiplier, as the following examples show.

EXAMPLE 1. Solve $\dfrac{3}{x+2} + \dfrac{x^2+11}{3x+6} = \dfrac{4x}{x+2}$.

Using $3(x+2)$ as the multiplier, we get the equation

$$9 + x^2 + 11 = 12x \quad \text{or} \quad x^2 - 12x + 20 = 0$$

or $(x-10)(x-2) = 0$. The roots are $x = 10$ and $x = 2$.

Using $(x+2)(3x+6)$ as a multiplier, we get

$$9x + 18 + x^3 + 2x^2 + 11x + 22 = 12x^2 + 24x$$

or
$$x^3 - 10x^2 - 4x + 40 = 0$$

or
$$(x-10)(x-2)(x+2) = 0$$

This equation has the roots $x = 10$, $x = 2$, and *also* the root $x = -2$. But the root $x = -2$ does not check in the given equation as it makes the denominator *zero* and zero may not be used as a divisor. The extra root was introduced by the unnecessary factor in the multiplier.

EXAMPLE 2. Solve $\dfrac{4x-12}{x-3} + \dfrac{6}{x-2} = 5$.

Using $(x-3)(x-2)$ as the multiplier, we get

$$4x^2 - 20x + 24 + 6x - 18 = 5x^2 - 25x + 30$$

or $x^2 - 11x + 24 = 0$ whose roots are $x = 8$ and $x = 3$.

However, $x = 3$ is not a root of the given equation as it makes the first denominator *zero*. Examining carefully the given equation we see that the first fraction, $\dfrac{4x-12}{x-3}$, is not in its simplest form. If we had first reduced this fraction to 4, our only answer would have been $x = 8$.

From these examples we learn that (1) the multiplier should not contain any unnecessary factors, and (2) each fraction in the equation should be in its simplest form before the L. C. M. of the denominators is selected. In general, the best rule is: *Always check the work.*

83. EXERCISES

The pupil should factor the denominators while copying the equations from the book. Exercises 1 to 17 will reduce to linear equations, and 18 to 31 to quadratic equations.

Solve the equations:

1. $\frac{2x+1}{5x-5} - \frac{x-4}{3x-3} = \frac{2}{x-1}$

2. $\frac{3y}{y-4} - \frac{y}{3y-12} = \frac{8}{4-y}$

3. $4 + \frac{2(r-11)}{2r+3} = \frac{5r}{r+3}$

4. $\frac{2t-3}{2t-2} = 1 - \frac{2-t}{1-t}$

5. $\frac{x-7}{2x-8} = \frac{x-3}{x} - \frac{1}{2}$

6. $\frac{y+1}{y+2} - \frac{y+6}{y+5} = \frac{-7}{2y+10}$

7. $\frac{5r-6}{r} = \frac{2r-33}{2r-3} + 4$

8. $\frac{t-1}{t-6} - \frac{t+8}{t+2} = \frac{1}{2t-12}$

9. $\frac{4}{x+1} - \frac{x^2-2x+2}{x^2-2x-3} = \frac{x}{3-x}$

10. $\frac{5}{3x} = \frac{x-5}{x^2-4x}$

11. $\frac{y}{3y+6} = \frac{y+2}{3y-6}$

12. $\frac{3r+2}{r-3} = \frac{8r+10}{3r-9}$

13. $\frac{2y-5}{2y-3} = \frac{2y-3}{2y}$

14. $\frac{r^2}{r^2-1} = \frac{3-2r}{2-2r}$

15. $\frac{t^2-2t}{t-2} = \frac{3}{5}$

16. $\frac{a+14}{4-a} = \frac{1-a}{a-9}$

17. $\frac{3r+22}{5r-10} = \frac{3r+2}{5r-20}$

Solve the equations:

18. $\dfrac{x-2}{x-1}+\dfrac{x-4}{x+1}=\dfrac{22}{x^2-1}$

19. $\dfrac{2x-9}{x-1}-\dfrac{x-4}{x}=\dfrac{1}{x^2-x}$

20. $\dfrac{2x}{x-1}+\dfrac{3-2x}{2x+2}=\dfrac{4x+9}{x^2-1}$

21. $\dfrac{2}{y}-\dfrac{3}{y+5}=\dfrac{3}{2}$

22. $\dfrac{t}{t+2}+\dfrac{t}{t^2-4}=\dfrac{6}{5}$

23. $\dfrac{r+3}{r+1}-\dfrac{r+2}{r+7}=2$

24. $\dfrac{x+4}{x}-\dfrac{x+6}{x+3}=\dfrac{1}{3}$

25. $\dfrac{y-6}{y}-\dfrac{y-8}{y+4}=\dfrac{1}{4}$

26. $\dfrac{2t+5}{2t+8}-\dfrac{1}{2}=\dfrac{t+4}{4t}$

27. $\dfrac{r}{r+3}+\dfrac{r-1}{2r+3}=1$

28. $\dfrac{a-6}{a-3}+\dfrac{7-a}{2a-9}=\dfrac{1}{3}$

29. $\dfrac{5}{s+2}-\dfrac{3}{s-2}=\dfrac{s-33}{s^2+2s}$

30. $\dfrac{18}{x}-\dfrac{12}{x-1}=\dfrac{1}{x-2}$

31. $\dfrac{6}{r-2}+\dfrac{4}{r-8}=\dfrac{2}{3}$

84. Equations Containing Decimals. When solving the equation $3.16\,x = 4.542$, the division of 4.542 by 3.16 gives $1.4373\cdots$. If the quotient is required to *the nearest tenth,* we say that $x = 1.4$ because 1.43 is nearer to 1.4 than to 1.5. Likewise, to *the nearest hundredth,* $x = 1.44$ because 1.437 is nearer to 1.44 than it is to 1.43; and to *the nearest thousandth,* $x = 1.437$ because 1.4373 is nearer to 1.437 than it is to 1.438. We notice, then, that the division must always be carried out one step farther than the last figure that we wish to retain.

To solve an equation like $\frac{7 - .3\,x}{.4} - \frac{6 - .25\,x}{.5} = 3.5$ we may either (1) use a multiplier, as 2 (the L. C. M. of .4 and .5), to clear of fractions, or (2) first eliminate some of the decimals.

Thus, if we multiply the numerator and the denominator of the first fraction by 10, and the numerator and the denominator of the second fraction by 4, we get

$$\frac{70 - 3\,x}{4} - \frac{24 - x}{2} = 3.5$$

85. EXERCISES

Find, to the nearest thousandth, the roots of:

1. $.257\,x = .163$
2. $.159\,y = .601$
3. $.2(.3\,x + .07) + 5(.188\,x - .2032) = .35$
4. $3(y - .82) = 1.19 - 1.53\,y$
5. $\frac{x + .25}{.2\,x + 1.15} = \frac{2\,x + 8.75}{.6\,x + 3.45}$
6. $\frac{5\,y - .9}{.3} + \frac{1.6 - 3\,y}{2} = \frac{2.6 - 8\,y}{1.2}$
7. $\frac{2\,r - .3}{.5} + \frac{r - .03}{.6} = \frac{5\,r - .69}{.3}$
8. $\frac{t - .6}{.3} + \frac{2\,t - .5}{.4} = \frac{t - .09}{.02}$

86. Problems Containing Literal Numbers. Solve the following problems:

1. The difference of two numbers is 13. Find the numbers if their sum is 51.

2. Find two numbers whose difference is 15 and sum 29.

3. Find two numbers one of which exceeds the other by 23, their sum being 117.

4. If one number is 30 larger than another and their sum is 92, what are the numbers?

These four problems are alike because in each one we are looking for two numbers whose *sum* and *difference* is known.

Hence let us call their sum s and their difference d. In the first problem $d = 13$ and $s = 51$. What are the values of d and s in ex. 2, 3, and 4?

If we examine the answers to each problem, we shall find that the larger of the two unknown numbers is always *one half the sum of s and d*, and that the smaller is always *one half the difference of s and d*. In other words:

Larger number $= \frac{1}{2}(s + d)$ Smaller number $= \frac{1}{2}(s - d)$

We can prove that these rules are correct by solving one of the problems, using the literal numbers s and d.

Let the larger number $= n$

Then the smaller number $= n - d$

Since the sum of the numbers is s, the equation is

$$n + n - d = s$$

Transpose $-d$ to the right: $2n = s + d$

Divide both members by 2: $n = \frac{1}{2}(s + d)$, the larger number

Also, $n - d = \frac{1}{2}(s + d) - d = \frac{1}{2}(s - d)$, the smaller number

The work above is the solution of the *general problem:*

Find two numbers whose sum is **s** *and whose difference is* **d.**

This problem differs from ex. 1 to 4 above in that specific numbers like 13, 51, etc. have been replaced by the literal numbers s and d.

87. EXERCISES

1. Prove the rules on page 78 by letting the smaller number equal n and the larger number equal $n + d$.

2. Show that if a field is 3 times as long as it is wide, then the width of the field $= \frac{1}{8}$ of the perimeter. (Let the width $= w$; then the length $= 3w$; perimeter $= p$. Show by an equation that $w = \frac{1}{8}p$.)

Write the statement in ex. 3 to 10 as equations, calling n the number which is asked for:

3. What number increased by 5 equals b?

4. What number increased by a equals b?

5. From what number must c be subtracted to give d?

6. Adding c to k times a number gives r.

7. If k times a certain number is subtracted from a, the result equals r.

8. Three times a number exceeds a by t.

9. Six times the sum of a number and b equals c.

10. The sum of 6 times a number and b equals c. (Notice the difference in the statements in ex. 9 and 10.)

11. Ask a classmate to choose some number and then perform in succession the following operations: (a) Multiply it by 10, (b) add 4, (c) multiply the result by 2, (d) add 3, (e) multiply the result by 5. If from this product you subtract 55 and divide the remainder by 100, you will have the original number.

Repeat this entire process, using the letter n to stand for the chosen number.

12. As in ex. 11, think of some number and then perform the following operations: (1) Multiply the number by a, (2) divide the product by b, (3) multiply the quotient by c, (4) divide the result by d, (5) multiply the quotient by $\frac{bd}{ac}$ and you will have the original number.

Repeat the process, choosing n as the original number.

88. Solving Literal Equations. To solve the equation $3x - ac = c^2 - ax$ for x, we first write the equation as $3x + ax = c^2 + ac$. If now the left member were $3x + 5x$, we could add these terms, getting $8x$; but how shall we add such terms as $3x + ax$?

If the question is asked: "By what numbers is x multiplied in the expression $3x + ax$?" the answer is:

"By 3 and by a; that is, by $(3 + a)$."

Hence $3x + ax = (3 + a)x$ as we have previously learned in *factoring*.

Study the following examples until you are able to tell in class what is *added* or *divided* in each step.

1. Solve $ax = b$.

$$x = \frac{b}{a}$$

Since x is multiplied by a, we find x by dividing both members by a.

2. Solve $a + x = b$.

$$x = b - a$$

Since a and x are added in the left member, we find x by transposing a to the right member.

Why do we transpose a in this equation, but divide by a in example 1?

3. Solve $2bx + c = 3d$.

$$2bx = 3d - c$$

$$x = \frac{3d - c}{2b}$$

The right member cannot be *actually* divided by $2b$, but we can *indicate* the division.

4. Solve $5x - a = 3bx + c$.

$$5x - 3bx = a + c$$

$$x(5 - 3b) = a + c$$

$$x = \frac{a + c}{5 - 3b}$$

How is the third equation obtained from the second? What is next done to both members?

89. EXERCISES

Solve for x, expressing the results in the simplest form:

1. $cx = d$
2. $r + x = s$
3. $a + bx = c$
4. $c - ax = b$
5. $(a + b)x = c$
6. $(a + x)r = s$
7. $ax + b = cx + d$
8. $ax - a = bx + b$
9. $3x - 9 = ax - a^2$
10. $ax - a^2 = bx - ab$
11. $ax - a^2 = 5a - 5x$
12. $bx - 3(x + a) = 2b$
13. $2(3ax - b) = 5(x + a)$
14. $5cx = 3d - 5bx + 12d$
15. $5bx - 3d = 2bx + 12d$
16. $a(x - 4) = a^2 - 2x + 4$
17. $ax - 5a + 3x = a^2 + 6$
18. $a(x + 3) = a^2 + 5x - 10$
19. $b(x - a) = b^2 + 2ab$
20. $b + a(x - b) = (b + a)x$

After clearing of fractions, solve for x or for y:

Review pages 28 and 74 to see how the multiplier is to be chosen.

21. $\frac{x}{6} - \frac{a}{3} = \frac{b}{4}$
22. $\frac{x}{a} - \frac{2}{b} = 3$
23. $\frac{2y}{9} - \frac{a}{4} = \frac{b}{6}$
24. $\frac{2y}{9} + \frac{a}{8} = \frac{a}{6}$
25. $\frac{a}{x} + \frac{b}{x} = c$
26. $\frac{x}{a} + \frac{x}{b} = c$
27. $\frac{x}{a} + \frac{x}{b} = 1$
28. $\frac{y}{a} - \frac{1}{ab} = \frac{y}{b}$
29. $\frac{x}{10} + \frac{x - b}{20} = b$
30. $\frac{x}{10} - \frac{x - b}{20} = b$
31. $\frac{b}{5} - \frac{2(y - b)}{3} = \frac{b}{10}$
32. $a - \frac{b}{y} = \frac{c}{2y}$
33. $\frac{b}{a} - \frac{a}{b} = \frac{b^2}{ax}$
34. $\frac{4a}{3} + 4 = \frac{a^2 + 3a}{3y}$
35. $\frac{x + a}{x} = \frac{x}{x - b}$
36. $\frac{a + 3}{ax - 3x} = \frac{1}{a - 3}$

Solve the following equations for x or for y:

37. $\frac{a+b}{1+x} = \frac{a-b}{1-x}$

38. $\frac{a}{y-b} = \frac{b}{y-a}$

39. $\frac{1}{a} + \frac{a}{x+a} = \frac{x-a}{ax}$

40. $\frac{3}{a} + \frac{y}{a-b} = \frac{3}{b}$

41. $\frac{3x}{50} + \frac{x+a}{20} = b$

42. $\frac{3x}{50} - \frac{x+a}{20} = b$

43. $\frac{a+b}{y} = \frac{c}{y+b}$

44. $\frac{1}{2} = \frac{y-1}{a} + \frac{2y-1}{2a^2}$

45. $\frac{y}{a} - \frac{1}{a+b} = \frac{y}{b} - \frac{1}{a-b}$

46. $\frac{x-a}{bc} + \frac{x-b}{ac} + \frac{x-c}{ab} = 0$

The following equations are quadratic, and are solved by factoring (see page 57). In solving a linear equation, the terms that contain x are collected in one member. In solving a quadratic equation, however, *all* the terms are collected in one member, making the other member zero.

47. $x^2 - 2ax - 15a^2 = 0$ ANS. $x = 5a$, and $x = -3a$

48. $y^2 + 3by = 10b^2$

49. $2x^2 + 5ax - 12a^2 = 0$

50. $2y^2 - 15by + 18b^2 = 0$

51. $10x^2 + 9b^2 = 21bx$

52. $8a^2x^2 - 14abx + 3b^2 = 0$

53. $x^2 + 3ax - 5bx - 15ab = 0$

54. $2y^2 - 2ay + 3by - 3ab = 0$

55. $y^2 - 3ay + 4by = 12ab$

56. $2x^2 - 2ax + ac = cx$

57. $(2x + b)(x - 2b) = 7b^2$

58. $(x + a)(x - 2a) - 4a^2 = 0$

59. $x^2 + 6ab = x(3a + 2b)$

60. $y^2 + 2ay - 2by - 8(a - b)^2 = 0$

90. EXERCISES — FORMULAS

1. After solving the equation $A = lw$ for l, translate the solution into a rule telling how the length of a rectangle can be found when the area and the width are known.

2. Solve $A = \frac{1}{2}bh$ for h and, as in ex. 1, state the solution as a rule telling how the altitude of a triangle can be found when the area and the base are known.

3. Solve $A = P + Prt$ for P and then state a rule for finding the principal when the amount, the rate, and the number of years are known.

4. Use the formula found in ex. 3 to find P when:

(a) $A = 1125,\ r = 5\%,\ t = 2\frac{1}{2}$

(b) $A = 3500,\ r = 6\%,\ t = 2$

The following formulas are taken from textbooks on algebra, physics, banking, engineering, etc.:

5. $S = \dfrac{n(a + l)}{2}$ Solve for a; for l.

6. $f = \dfrac{mg - t}{m}$ Solve for m.

7. $R = \dfrac{WL - a}{L}$ Solve for L.

8. $u = \left(\dfrac{m}{M + m}\right)v$ Solve for m.

9. $C = \dfrac{E}{R + \dfrac{r}{n}}$ Solve for n; for r.

10. $T = \dfrac{C + \dfrac{C}{n}}{t}$ Solve for C.

11. $L = \dfrac{Wh}{d(W - P)}$ Solve for W.

12. $P = \dfrac{2400}{D}(3T - 1) - 76$ Solve for T.

91. Suggestions for Solving Problems. The pupil should refer to this page for help whenever he has difficulty with problems of the kind shown on page 85.

1. In many problems dealing with literal numbers it is not necessary to use an equation or to represent the unknown quantity by any letter. For example:

If a dealer buys pencils for c cents a dozen and sells them for s cents each, what is the gain in dollars on n dozen?

Find the cost of one pencil, then the profit on one pencil; next find the profit on one dozen, and finally the profit on n dozen. This profit may then be expressed in dollars.

A dealer buys goods for b dollars and sells them for s dollars. What is the per cent of gain on the cost?

First find the gain in dollars. The per cent of gain is 100 times the quotient of the gain by the cost.

2. Proportions may well be used in such problems as:

If n yards of goods cost c cents, find the cost of m yards.

Here the ratio of the costs is equal to the ratio of the amounts purchased. Represent the unknown cost by x, write the proportion, and solve it for x.

If it takes m men h hours to do some work, how long will it take n men to do the same work?

In this case the number of men is *inversely* proportional to the number of hours.

3. Many problems are like those on pages 33 and 34 with literal numbers substituted for the usual numbers, as:

One man can do certain work in a days; another man can do the same work in b days. How many days will it take if both men work?

If the answer is x days, then $\frac{x}{a} + \frac{x}{b} = 1$.

Find three consecutive integers whose sum is s.

If n is the first of the 3 numbers, then $n + (n + 1) + (n + 2) = s$.

92. PROBLEMS WITH LITERAL NUMBERS

1. A man buys a house for b dollars and sells it at a profit of $r\%$. What is the selling price?

2. If a train goes m miles in h hours, how many feet does it go in a second? how many yards in a minute?

3. What per cent is gained on the cost if an article is bought for b dollars and sold for s dollars? What per cent is gained on the selling price?

4. If m men can finish a job in d days, how many more men should be hired if the work must be done in c days?

5. A can do certain work in a days and B in b days. After A has worked alone for c days, he hires B to help him. In how many more days will the work be finished?

6. A dealer mixes a pounds of candy worth b cents a pound with c pounds worth d cents a pound. What is the value of the mixture per pound?

7. Divide d dollars among 3 brothers so that each one shall have k more dollars than the next younger.

8. A man invests s dollars, part at 6% and the remainder at 5%. If the annual income is i dollars, how much does he invest at each rate?

9. Find the amount of water that must be added to one ounce of $r\%$ sugar solution to make a $t\%$ solution.

10. If a father is f years old when his son is s years old, in how many years will the father be k times as old as his son?

11. The arms of a lever are l feet and r feet, and weights of c pounds and d pounds, respectively, are placed at the ends. Write the equation for l, r, c, and d.

12. What is the average selling price of an article if a articles are sold at r cents each, b at s cents each, and c at t cents each? If the cost of the articles was k cents each, what is the total profit?

93. REVIEW OF CHAPTERS II TO V

1. For building a concrete road, the county contributes 50% as much as the state, and the town pays 40% as much as the county. If the cost per mile is $34,000, find the amount contributed per mile by the state, the county, and the town.

2. In a solution of 80 lb. of salt and water there are 4 lb. of salt (that is, 5% is salt). How many pounds of water must be evaporated to make the new solution 12% salt?

3. Glenn and Chester, weighing 80 lb. and 90 lb. respectively, sit 5 ft. and 3 ft. from the fulcrum on the same side of a seesaw. How far from the fulcrum, on the other side, must Kenneth, weighing 120 lb., sit in order to balance Glenn and Chester?

4. If $x = 5$ is a root of the equation $3x^2 - 13kx = 10$, what is the value of k? (If $x = 5$ is a root, then 5 may be substituted for x in the equation.)

5. If $x - 3$ is a factor of the quantity $2x^2 + kx - 33$, what is the value of k? (If $x - 3$ is a factor, then the quantity must reduce to zero when 3 is substituted for x.)

Use the suggestion in ex. 5 to work ex. 6 and 7.

6. If $x - 4$ is a factor of $kx^2 - 11x + 12$, what is the value of k?

7. What is the value of k if $3x^2 + 4kx - 14$ is divisible by $x + 2$ without a remainder?

8. In the formula $V = 2gx^2\left(\frac{1}{x} - \frac{1}{y}\right)$ substitute $x = y - a$ and simplify the result as far as possible.

9. Find the value of $x - \frac{1}{x}$ when $x = \frac{r - s}{r + s}$.

10. Prove by substitution that $x = \frac{ab}{a + b}$ is a solution of $\frac{x}{ab} - \frac{1}{a + b} = x(a + b) - ab$.

11. Solve the equation $\frac{8}{x^2 - 3x + 2} = \frac{7}{x^2 - 4x + 3}$ by regarding it as a proportion and then explain why one of the values of x does not check. (Remember that the product of the means equals the product of the extremes.)

Solve the equations:

12. $\dfrac{3 + \frac{4}{y}}{2} + \dfrac{5 - \frac{2}{y}}{3} = 3 + \frac{2}{y}$

13. $\frac{4n + 5}{n + 2} - 4 = \frac{n}{n^2 + 4n + 4}$

14. $\frac{x - 1}{x} - \frac{x + 5}{x + 2} = \frac{-9}{2x + 4}$

15. $\frac{15t - 2}{3t - 2} = 6 - \frac{3t - 10}{3t - 6}$

16. $\frac{x - a}{a} + \frac{x - b}{b} = \frac{c - x}{c} - 3$

17. $\frac{a}{x - a} - \frac{b}{x - b} + \frac{b}{x + c} = \frac{a}{x + c}$

18. $\frac{x - s}{r - s} + \frac{x + s}{r + s} = \frac{2sx}{r^2 - s^2}$

19. $\frac{a + b}{y} = \frac{c}{y + b}$

20. $\frac{a}{x} = \frac{b^2 - bx}{ax - x^2}$

21. $\frac{1 - ax}{bx} = \frac{bx - 1}{ax}$

22. $\frac{bx}{a} = \frac{a^2 - ax}{ab - bx}$

23. Simplify $\frac{1 - a^2}{1 + b} \cdot \frac{1 - b^2}{a + a^2}\left(1 + \frac{a}{1 - a}\right)$

24. Simplify $\dfrac{\frac{a^2 + b^2}{a} - b}{\frac{1}{a} - \frac{1}{b}} \div \dfrac{a^3 + b^3}{a^3 - b^3}$

94. PROBLEMS FROM VARIOUS SCIENCES

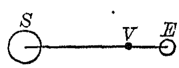
1. The speed at which sound travels in air is given by the formula $v = 1090 + 1.14(t - 32)$ where v is the speed in feet per second and t is the Fahrenheit temperature.

Find (*a*) the speed when the temperature is 68°, and (*b*) the temperature when the speed is 1100 ft. per second.

2. Two men ride in the same direction around a circular path, one making the circuit in $2\frac{1}{2}$ hr., and the other in $3\frac{3}{4}$ hr. If they start together, after how many hours will they be together again; that is, how soon will one man make one more circuit than the other?

Ex. 2 is like the astronomical problem in ex. 3 and 4.

3. If a planet is between the earth and the sun, as in the figure, the planet is said to be *in inferior conjunction* with the earth. Venus was in conjunction with the earth on Feb. 7, 1926. If Venus and the earth revolve about the sun in 225 days and 365 days respectively, find the approximate date of the next inferior conjunction.



4. Mercury revolves around the sun in about 88 days. If Mercury and the earth were in inferior conjunction on Dec. 11, 1925, find the approximate date of the next inferior conjunction.

5. In business the "turnover" represents the value of the goods sold which the dealer must repurchase from his wholesaler in order to keep his stock up to date. The "turnover," T, in dollars, should be $T = \frac{1}{t}\left(C + \frac{C}{k}\right)$, where C is the capital invested in the business, t is the number of months required to make the sales, and k is a certain number depending on the kind of business.

(*a*) Find T if $C = 50{,}000$, $t = 2$, $k = 1.5$.
(*b*) Find C if $T = 6000$, $t = 3$, $k = 2$.

CHAPTER VI

GRAPHS. FUNCTIONS

95. Graphs of Statistics. Graphs are pictures that show how a quantity changes from time to time or how it depends on some other quantity. Fig. 1 below shows the number of touchdowns made by a football team during five seasons. Fig. 2 shows the weight in pounds of a boy at various ages.

YEAR	TOUCH-DOWNS
1	15
2	12
3	10
4	18
5	21

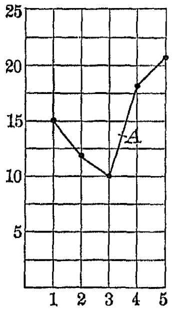


FIG. 1

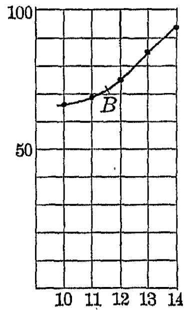


FIG. 2

AGE	WEIGHT
10	67
11	69
12	75
13	84
14	93

The graph is the broken or curved line joining the points marked with dots.

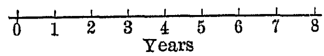
Notice this difference in the two graphs: The point A in the first graph does not represent any number of touchdowns; but the point B in the second graph does represent a weight. (It is the weight of the boy when $11\frac{1}{2}$ years old.) The only points on the first graph that have any meaning are those marked with dots; but in the second graph, the whole curve has a meaning. For this reason the first graph is made of broken lines, while the second is drawn as a smooth curve.

96. Construction of Graphs. The successive steps in the drawing of a graph are shown in the following problem:

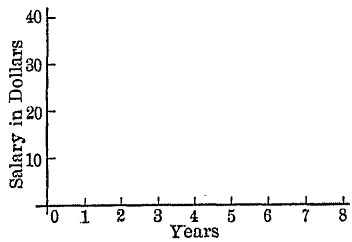
A boy's weekly wage for various years after graduation from high school was:

YEAR	1	2	3	4	5	6	7	8
WAGE	$16	$17	$18	$20	$24	$30	$33	$34

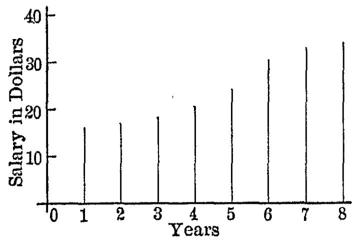
1. Draw a horizontal scale, or axis.



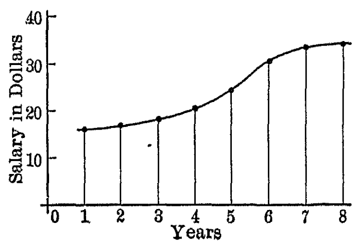
2. Through the zero point draw a vertical scale. The numbers on it must be large enough to show the largest wage, but the divisions on the vertical scale need not be the same as on the horizontal scale.



3. At point 1, draw a line to represent $16, as that is the weekly wage during the first year. Similarly, draw the vertical lines at the points 2, 3, etc.



4. Join the tops of the vertical bars by a smooth curve or a series of lines.



If paper with vertical and horizontal lines on it is used, the vertical bars shown in step 3 need not be drawn. Use dots to mark the points where the ends of the bars would be.

97. EXERCISES

Draw the graphs for the following data. State on the axes exactly what the numbers represent, as "years," "cost per pupil," "number of members," etc.

1. The cost per pupil of maintaining the high schools in a certain city was:

Year	1918	1919	1920	1921	1922	1923	1924
Cost per pupil . .	\$27	\$35	\$40	\$42	\$43	\$44	\$45

2. The cost of a certain kind of automobile varied from year to year as follows:

Year	1919	1920	1921	1922	1923	1924	1925
Cost	\$900	\$950	\$875	\$800	\$850	\$875	\$850

In ex. 2, the prices vary between 800 and 1000; hence we may mark the lowest number on the vertical scale as 800 instead of 0.

3. The membership in a civics club in a school grew as follows:

Year of club	1	2	3	4	5	6	7
Number of members . . .	600	620	680	750	840	870	910

4. When Mr. Smith traveled across the country, the mileage and the consumption of gasoline increased as follows:

Miles	0	100	200	300	400	500	600	700
Gallons of gasoline . .	0	5	12	20	28	35	41	50

During what part of his trip were the roads the best?

5. As Mr. Jones's salary increased, he saved each year a larger per cent of his salary, as shown in the table:

Year	1	2	3	4	5	6	7	8
Part of salary saved	8%	9%	10%	12%	15%	16%	17%	18%

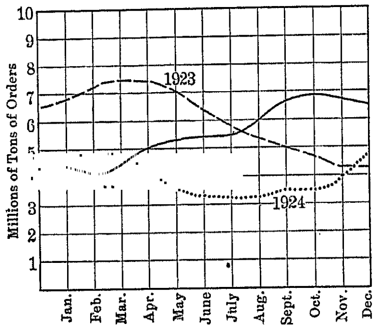
6. A man 25 yr. old may have his life insured for \$1000 by making a yearly payment of \$20, called a *premium*. The premium varies with the age when insured as follows:

Age in years	25	30	35	40	45	50	55
Premium	\$20.00	\$23.00	\$26.50	\$31.00	\$38.00	\$46.50	\$59.00

98. Interpretation of Graphs. When several graphs are drawn on the same axes, we can better understand the meaning of the rise and fall of the curve.

EXAMPLE. The number of millions of tons of orders of a steel corporation on the last day of each month during 1922, 1923, and 1924 is shown in the table below. The figure shows the three graphs, one for each year.

	1922	1923	1924
JAN.	4.4	6.9	4.8
FEB.	4.1	7.3	4.9
MAR.	4.5	7.4	4.8
APR.	5.1	7.3	4.2
MAY	5.3	7.0	3.6
JUNE	5.6	6.4	3.3
JULY	5.6	5.9	3.2
AUG.	6.0	5.4	3.3
SEPT.	6.7	5.0	3.5
OCT.	6.9	4.7	3.5
NOV.	6.8	4.4	4.0
DEC.	6.7	4.4	4.8



Business of all kinds was reported as "very good" in 1922, "dull" in 1923, and "improving" after August, 1924. According to the above graphs, how did the steel business vary in these years? Why is the manufacture of steel called a "barometer" of the country's prosperity?

The production of crude oil for gasoline varies with the season, being greatest in the spring and summer months. Hence it is called a "seasonal" business. Judging from the above graphs, is the manufacture of steel a seasonal business?

When constructing a graph, state very exactly in the figure what the numbers on the two scales or axes represent. After the graph is constructed, aim to *interpret* it; that is, observe and try to explain the high and low points and the variation of the curve.

99. EXERCISES

Draw the graphs to represent the following data:

1. The approximate income and expenses of a drug store for a year are shown below. Draw both graphs on the same set of axes.

	Income	Expenses		Income	Expenses
Jan.	$10,000	$8,000	July	$16,000	$12,000
Feb.	12,000	9,000	Aug.	18,000	13,500
Mar.	13,000	9,500	Sept.	17,000	13,000
Apr.	12,000	9,200	Oct.	13,000	10,000
May	11,000	8,400	Nov.	11,000	12,500
June	13,000	9,200	Dec.	15,000	11,000

2. The following table gives the amount (the sum of the principal and the interest) of $1 at 5% simple interest and also at 5% interest compounded annually. Draw both graphs on the same set of axes.

Years	5	10	15	20	25
Amount at simple interest . .	$1.25	$1.50	$1.75	$2.00	$2.25
Amount at compound interest .	$1.28	$1.63	$2.08	$2.65	$3.39

Draw the graphs for ex. 3 and 4 on one set of axes.

3. The value in hundreds of millions of dollars of the exports from the United States for 1924 was:

Jan.	3.9	Apr.	3.5	July	2.8	Oct.	5.3
Feb.	3.7	May	3.4	Aug.	3.3	Nov.	5.1
Mar.	3.4	June	3.1	Sept.	4.3	Dec.	5.0

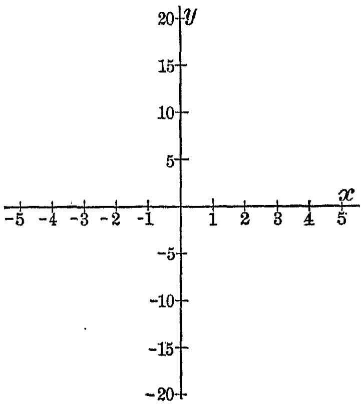
4. The value in hundreds of millions of dollars of the imports to the United States for 1924 was:

Jan.	3.0	Apr.	3.2	July	2.8	Oct.	3.1
Feb.	3.3	May	3.0	Aug.	2.5	Nov.	3.0
Mar.	3.2	June	2.9	Sept.	2.9	Dec.	3.0

Judging from the remarks in the example on page 92, how do exports and imports vary during good and bad times?

100. Graphs Showing Negative Numbers. The table below contains negative as well as positive values of the two quantities x and y:

x	-4	-3	-2	-1	0	1	2	3	4
y	16	8	5	4	3	2	-4	-14	-18

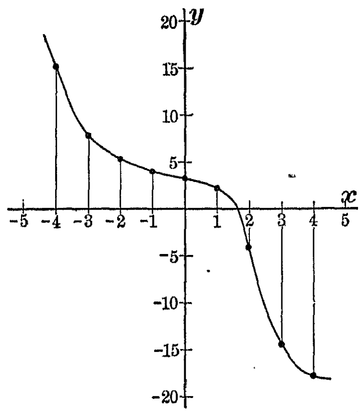


First draw two scales, as in the figure at the left. On the horizontal scale the values of x increase from left to right; on the vertical scale the values of y increase from the bottom upwards. The distance from 0 to 1 on the horizontal scale need not be the same as the distance from 0 to 1 on the vertical scale.

When $x = -4$, the value of y is 16; hence at the point -4 on the horizontal scale draw a line upwards to represent 16.

When $x = -3$, the value of y is 8; hence at the point -3 draw a line upwards to represent 8.

Continue in this way for all the values of x. When $x = 2$, the value of y is -4, and so the line is drawn *downwards* instead of upwards.



101. EXERCISES

Draw the graphs for the following tables:

1.

x	-4	-3	-2	-1	0	1	2	3	4
y	-7	-5	-3	-1	1	3	5	7	9

2.

x	-3	-2	-1	0	1	2	3
y	-10	-3	2	5	6	5	2

3.

x	-3	-2	-1	0	1	2	3
y	11	8	5	2	-1	-4	-7

4.

x	-3	-2	-1	0	1	2	3	4
y	-32	0	12	10	0	-12	-20	-18

5.

x	-10	-5	0	5	10	15	20
y	45	30	15	0	-15	-30	-45

6. In ex. 1, 3, 5, the graphs are straight lines. What do you notice about the successive values of y in these tables?

102. EXERCISES — GRAPHS OF EQUATIONS

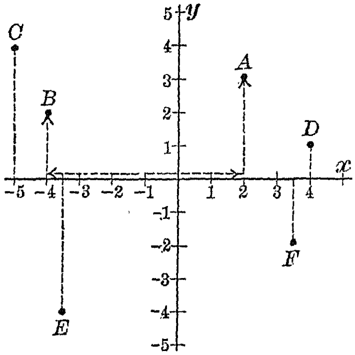
1. If $y = 3x + 6$, find the values of y when x has the values $-4, -3, -2, -1, 0, 1, 2, 3, 4$. Write the values of x and y in a table and then draw the graph.

In the same way draw the graphs for the following equations, using values of x from -4 to $+4$:

2. $y = x + 3$
3. $y = 3 - x$
4. $y = x - 3$
5. $3y = x + 4$
6. $3y + 2x = 6$
7. $4y - 6x = 9$
8. Draw the graph for $y = x$.
9. On the same set of axes draw the graphs for both $y = x + 2$ and $y = -x + 2$.
10. On the same set of axes draw the graphs for both $y = x + 4$ and $y = x - 4$.
11. On the same set of axes draw the graphs for both $y = 3x + 2$ and $y = -\frac{1}{3}x + 2$.

103. Definitions of Terms. The horizontal line on which the values of x are marked is called the x *axis*. The vertical line, or, more correctly, the line perpendicular to the x axis, is called the y *axis*. Frequently we give the axes special names, such as the *time axis* or the *temperature axis*. The point where the axes cross is called the *origin*.

The location of any point in the figure can be stated by telling the value of x and of y at that point. This is the same as telling how far the point is to the right or left of the vertical axis, and how far it is above or below the horizontal axis. Thus, for point A, $x = 2$, $y = 3$; for point B, $x = -4$, $y = 2$. The arrows show how to get to the point if we start from the origin.



The two numbers which locate a point are called the *coördinates* of the point.

The number x is called the *abscissa* of the point, and the number y is called the *ordinate* of the point.

Frequently instead of saying $x = 3$, $y = 5$ for a point, we say merely "the point (3, 5)," it being understood that the first number in the parenthesis is the value of x and the second number the value of y. What are the coördinates of the points C, D, E, F in the figure above?

Exercise. Draw an x axis and a y axis. On the x axis show numbers from -5 to $+5$, and on the y axis show numbers from -10 to $+10$. Then draw the arrows which locate the following points:

G, $(3, -4)$	I, $(-3, 8)$	K, $(-2, 5)$
H, $(-3, -8)$	J, $(0, 5)$	L, $(5, 0)$

104. Variables. If s represents the length of a side of a square and p the length of the perimeter, then $p = 4s$. In this equation the letter s does not denote some fixed length but the length of a side of *any* square; likewise, p denotes the perimeter of *any* square.

A letter that represents any number that can be used in a problem is called a variable number, or a *variable*.

The number 4 in $p = 4s$ does not vary, because every square has 4 sides. The number 4 is constant while the numbers p and s change or vary in different squares.

In the equation $p = 4s$ we have the variables p and s, and the constant 4. The equation tells how the variable p depends on the variable s, or expresses p *in terms of* s. If we solve the equation for s, writing $s = \frac{1}{4}p$, then the side is expressed *in terms of* the perimeter.

Similarly, the equation $C = 2\pi r$ contains the variables C and r, and the constants 2 and π; and the circumference is expressed *in terms of* the radius. If written as $r = \frac{C}{2\pi}$, the radius is expressed in terms of the circumference.

105. ORAL EXERCISES

State the variables and the constants in the following equations. Solve each equation for each variable, stating what variable is then expressed in terms of what other variables.

1. $F = \frac{9}{5}C + 32$ (Fahrenheit and centigrade temperatures)
2. $A = \frac{1}{2}bh$ (Area of a triangle)
3. $P = 3s$ (Perimeter of an equilateral triangle)
4. $C = \pi d$ (Circumference and diameter of a circle)
5. $p = 2(l + w)$ (Perimeter of a rectangle)
6. $W = 62.5c$ (Weight of c cubic feet of water)

106. Functions. Instead of saying that the circumference of a circle *depends on* the radius, we frequently say that the circumference is a *function* of the radius.

In an equation containing several variables, one of the variables is called a function of the other variables.

It is just as correct to say that the radius of a circle is a function of the circumference as to say that the circumference is a function of the radius. We use one expression or the other, according to the nature of the problem. If we give different values to r and then find the value of C, we are thinking of the circumference as depending on the radius, and hence say that C is a function of r. If, however, we give different values to C and find the values of r, then we are thinking of r as depending on C, and in this case r is a function of C.

107. The value of any quantity containing x, as $3x - 5$, depends on the value of x. Thus, $3x - 5$ equals -2 when $x = 1$; equals 4 when $x = 3$; equals 13 when $x = 6$, etc. Since x may be given any value, x is a variable; and the quantity $3x - 5$ is a function of x because its value depends on the value of x.

Any expression containing x is a function of x.

Instead of denoting $3x - 5$ by y and writing $y = 3x - 5$, it is frequently convenient to use the symbol $f(x)$ for any function of x, writing: $f(x) = 3x - 5$, for example. This is read "the function of x is $3x - 5$." This notation is very useful when we wish to indicate that various numbers or letters are to be substituted for x. Thus, $f(a)$ means the result of substituting a for x. For example:

If $f(x) = x^2 - 3x + 5$, then $f(a) = a^2 - 3a + 5$, and $f(4) = 4^2 - 3 \cdot 4 + 5$ or 9, and $f(-6) = 36 + 18 + 5$ or 59.

If we are working with several functions, we may denote one of them by $f(x)$, another by $g(x)$, another by $F(x)$, etc.

108. EXERCISES ON THE FUNCTIONAL NOTATION

These exercises are useful as a drill on the vocabulary.

1. If $f(x) = 2x + 3$, find the value of $f(x)$ when $x = 1$; that is, find the value of $f(1)$.

Find the values of $f(2)$, $f(3)$, $f(-4)$, $f(\frac{1}{2})$, $f(\frac{1}{3})$.

2. Given $f(x) = 2x - 3$. Find the values of

$$f(-3),\ f(-2),\ f(-1),\ f(0),\ f(1),\ f(2),\ f(3).$$

Write the values of $f(x)$ in a table as shown at the right. Draw a graph from this table.

x	-3	-2	-1	0	etc.
$f(x)$					

This is the graph of the function $2x - 3$.

As in ex. 2, draw the graphs of the following functions, using values of x from -3 to $+3$:

3. $4x + 5$
4. $x^2 - 1$
5. $x^2 - 3x + 2$
6. $x^2 - 5x + 6$

7. Given $f(x) = x^2 - 3$. Does the graph of $f(x)$ differ from the graph of the equation $y = x^2 - 3$?

8. If $f(x) = x^2 - 5x + 6$, find:

(*a*) The values of x for which $f(x) = 0$.

(*b*) The values of x for which $f(x) = 20$.

9. Given $f(x) = x^2 - 3x + 5$, arrange the following expressions in order from the smallest to the largest:

$$f(\tfrac{1}{2}),\ f(-2),\ f(0),\ f(4),\ f(2),\ f(.6)$$

10. Fill in the blanks in the following statement of the Factor Theorem, stated at the bottom of page 52:

If $f(n) = 0$ then $\cdots$ is a factor of $f(x)$ provided $f(x)$ is a rational integral $\cdots$ of x.

11. Complete the following statement of the facts that were discovered in steps 1 to 4 on page 52:

If $f(x)$, a rational integral function of x, is divided by $x - n$, the remainder is $\cdots$.

109. Direct Variation. The following table shows some corresponding values of x and y when $y = kx$. The pupil should write out a similar table for $y = 3x$ or $y = -4x$.

x	-3	-2	-1	0	1	2	3	4	5
y	$-3k$	$-2k$	$-k$	0	k	$2k$	$3k$	$4k$	$5k$

The quotient of any value of y (except 0) by the corresponding value of x is always k, because $\frac{y}{x} = k$ if $y = kx$.

If $y = kx$ *then the values of* y *are proportional to the values of* x. This idea is also stated in such phrases as: y *is directly proportional to* x, or y *varies as* x.

The converse of this statement is also true; namely:

If one variable is proportional to another, or if y varies as x, then the relation between them is $y = kx$.

110. Inverse Variation. The following table for the equation $y = \frac{k}{x}$ shows another kind of variation:

x	-3	-2	-1	0	1	2	3	4	5
y	$-\frac{1}{3}k$	$-\frac{1}{2}k$	$-k$		k	$\frac{1}{2}k$	$\frac{1}{3}k$	$\frac{1}{4}k$	$\frac{1}{5}k$

(Why is there no value of y when $x = 0$? See page 10.)

In this table the product of x and y is always k, as the equation itself shows: $xy = k$ if $y = \frac{k}{x}$.

If $y = \frac{k}{x}$, *then* y *is said to be inversely proportional to* x, *or* y *varies inversely as* x; *and conversely.*

One variable may depend on another in a great many other ways than the two ways shown above. Thus, if $y = kx^2$, we say that y varies as the square of x; and if $y = \frac{k}{x^2}$, we say that y varies inversely as the square of x. When the kind of variation is not specified at all we say that y varies *with* x instead of y varies *as* x.

111. Denote any two of the values of x by x_1 and x_2 (read as "x sub-one" and "x sub-two"), and the corresponding values of y by y_1 and y_2. Then the equation $\frac{y_1}{y_2} = \frac{x_1}{x_2}$ is another way of stating that y varies as x and the equation $\frac{y_1}{y_2} = \frac{x_2}{x_1}$ is another way of stating that y varies inversely as x.

Whether this method or the methods on page 100 should be used depends on the nature of the problem.

EXAMPLE 1. y varies as x, and $y = 8$ when $x = 2$. Write the equation for y and x.

Since y varies as x, let $y = kx$. The value of k is then found by substituting $y = 8$, $x = 2$ in this equation. This gives $8 = k \cdot 2$, or $k = 4$. Hence $y = 4x$ is the desired equation.

EXAMPLE 2. y varies as x, and $y = 8$ when $x = 2$. What is the value of y when $x = 3$?

Using the method at the top of this page, write $\frac{8}{y_2} = \frac{2}{3}$. Hence $y_2 = 12$; that is, $y = 12$ when $x = 3$.

This value of y could also be found by substituting $x = 3$ in the equation $y = 4x$, found in example 1.

112. EXERCISES

Write the equation for y and x in ex. 1 to 3:

1. y varies as x, and $y = 16$ when $x = 3$.
2. y varies inversely as x, and $y = 20$ when $x = 4$.
3. y varies as the square of x, and $y = 32$ when $x = 2$.
4. In ex. 1, find the value of y when $x = 5$.
5. In ex. 2, find the value of y when $x = 6$.
6. y varies as x, and $y = 12$ when $x = 3$. What is the value of y when $x = 5$?
7. y varies inversely as x, and $y = 100$ when $x = 5$. What is the value of y when $x = 50$?

113. EXERCISES

By studying the numbers in the following tables:

(*a*) Find whether y varies directly or inversely as x.

(*b*) Write the equation for x and y.

(*c*) Find the values of x and y for the blank spaces.

1.

x	3	4		6	7	8
y	12	16	20		28	

2.

x	2	4	8		32	
y	16	8		2		$\frac{1}{2}$

3.

x	1	2	3		5	
y	10	5		$2\frac{1}{2}$	2	$1\frac{1}{3}$

4.

x	-3	-2	-1	0		2
y	9	6	3		-3	

After introducing suitable letters for the variables, write the following statements as equations (*a*) as shown on page 100, and also (*b*) as shown in § 111, page 101.

5. The *volume* of a gas is inversely proportional to the *pressure* exerted on it.

6. The *volume* of a gas varies as its *temperature*.

7. The *area* of a rectangle varies as its *length*.

8. The *area* of a triangle varies as its *altitude*.

9. The electrical *resistance* in a wire varies as the *length* of the wire.

10. The electrical *current* varies inversely as the *resistance* in the circuit.

11. The length of the circumference of a circle varies as the radius.

12. The area of a circle varies as the square of the radius.

13. The number of revolutions of a wheel on a car varies inversely as the circumference of the wheel.

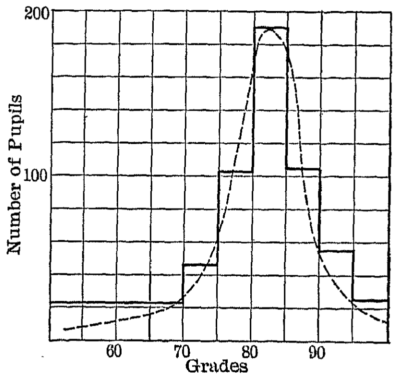
14. The volume of a cylinder varies as its height.

114. SUPPLEMENTARY EXERCISES — FREQUENCY GRAPHS

1. The following grades in algebra were made by the 544 freshmen in a school:

Grade . . .	0–69	70–74	75–79	80–84	85–89	90–94	95–100
Number of pupils	22	45	102	190	105	56	24

These facts are shown in the figure by the series of horizontal lines. If we counted how many pupils made each grade, as 75, 76, 77, etc., the graph would look more like the dotted curve.



This curve shows the distribution of the grades among the pupils, and is called a *frequency* graph.

Draw the frequency graphs for the following:

2. The weights of some 13-year-old boys are given below:

Weight in lb.	Less than 61	61–68	68–75	75–82	82–89	89–96	96–103	103 or more
Number of boys	15	34	75	85	62	37	14	11

(For example, 15 boys weighed less than 61 lb., 34 boys weighed 61 lb. or more but less than 68 lb.) The horizontal scale need not begin with 0; begin here with 54, and let each division be 7 lb. In your figure show both the series of lines and also the smooth curve that might result if 1 lb. divisions were used.

3. The lengths of some ears of corn taken from one garden varied as follows:

Length in inches . .	6–7	7–8	8–9	9–10	10–11	11–12
Number of ears . . .	50	150	300	250	125	25

115. Mathematics and Other Sciences. On page 98 it was stated that a function of x is some expression that involves the number x. The essential idea is that there are two variables, one of which depends on the other. There are many such relations in geometry and physics, some of which are stated in the exercises on page 102.

There are, however, a great many relations in other fields of study in which the dependence cannot be stated by means of an equation. Thus, the number of bushels of wheat obtained from a field depends on the rainfall, but we cannot express this dependence by means of an equation. We can represent the variation by a graph after measuring the wheat obtained and the rainfall, and sometimes it is possible to derive what is called an *empirical* (or experimental) formula.

Likewise, the economist states that the demand for an article depends on its price; the banker says that prosperity depends on credit; the chemist says that the speed of a reaction depends on the strength of the chemicals; and the doctor says that our health depends on the number of certain kinds of corpuscles in our blood. Every science is a study of the dependence of one variable on another.

Mathematics aims to state these relations in algebraic symbols and in equations. The study of the equation, then, frequently leads to new relations and new discoveries. Thus, mathematics is a tool used by all other sciences and the notion of function is fundamental to all. We may define it as follows:

One quantity is called a function of another quantity if to every value of the one there corresponds a value of the other.

CHAPTER VII

SETS OF LINEAR EQUATIONS

116. Review of Sets of Equations in Two Unknown Numbers. The following illustration shows how a problem may lead to a set of two equations in two unknown numbers.

Problem. The force P necessary to lift a weight W by means of a certain machine is given by the formula:

$$P = a + bW$$

By trial it is found that a force of 4 lb. will lift a weight of 10 lb. and a force of 6 lb. will lift 20 lb. Find the value of the constants a and b for the machine.

The information states that $P = 4$ when $W = 10$. If we substitute these values in the formula, we get:

$$4 = a + 10\,b \qquad (1)$$

The information also states that $P = 6$ when $W = 20$. Hence

$$6 = a + 20\,b \qquad (2)$$

We thus have two equations to solve for a and b.

We can find pairs of numbers that will satisfy one of these equations and not the other. Thus, $a = 3$, $b = \frac{1}{10}$ satisfies equation (1) but not (2); and $a = 1$, $b = \frac{1}{4}$ satisfies (2) but not (1). The pair of numbers that satisfies *both* equations is called *the solution* of the set.

The solution of equations (1) and (2) is $a = 2$, $b = \frac{1}{5}$.

In earlier work in algebra we learned three ways of solving *sets of equations* (sometimes called *simultaneous equations*); namely, by graphs, by the multiplication-addition method, and by the substitution method. We shall review these methods as a preparation for solving sets of three equations in three unknown numbers.

117. Review of Graphic Solution of a Set of Equations.

EXAMPLE. Solve graphically $\begin{cases} x + 2y = 3 \\ x - 2y = 2 \end{cases}$

First, make a table of values for each equation:

When making a table of values, it is convenient to solve the equations for y.

Thus, $x + 2y = 3$ is written $y = \frac{1}{2}(3 - x)$ and $x - 2y = 2$ is written $y = \frac{1}{2}(x - 2)$

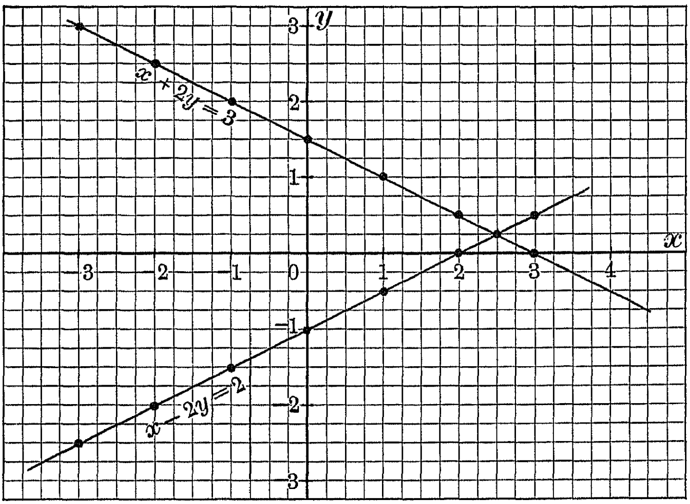
$x + 2y = 3$

x	y
-3	3
-2	2.5
-1	2
0	1.5
1	1
2	.5
3	0

$x - 2y = 2$

x	y
-3	-2.5
-2	-2
-1	-1.5
0	-1
1	$-.5$
2	0
3	.5

Second, draw on one set of axes the two graphs, as in the figure. The two lines cross at the point where $x = 2\frac{1}{2}$, $y = \frac{1}{4}$. This pair of numbers is the solution of the set.



An equation like $ax + by + c = 0$ is called a *linear equation* because its graph is a straight line.

118. Classification of Sets of Linear Equations. The set shown at the right has no solution because one equation states that $2x + 3y$ equals 5 and the other that $2x + 3y$ equals 7. The equations contradict each other. Such equations are called *inconsistent*. Their graphs are parallel lines, and hence the set has no solution.

$$\begin{aligned} 2x + 3y &= 5 \\ 2x + 3y &= 7 \end{aligned}$$

The next set has a different peculiarity. If we multiply the first equation by $\frac{3}{2}$ we obtain the second. Any pair of numbers that satisfies the first equation will satisfy the second. Such equations are called *dependent* because each one can be derived from the other. Graphically, both equations represent the same line.

$$\begin{aligned} 2x + 4y &= 5 \\ 3x + 6y &= 7\tfrac{1}{2} \end{aligned}$$

All other sets are called *independent*. Their graphs are distinct non-parallel lines.

If the equations are written with the general coefficients a, b, etc., it can be proved that the equations are:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

Independent if the quantity $ae - bd$ is not zero.
Either *dependent* or *inconsistent* if $ae - bd = 0$.

119. EXERCISES

Are the following sets *inconsistent*, *dependent*, or *independent?*

1. $\begin{cases} 6x - 4y = 3 \\ 3x - 2y = 1 \end{cases}$

2. $\begin{cases} 2x + 3y = 9 \\ 3x - 2y = 7 \end{cases}$

3. $\begin{cases} x + 2y = 2 \\ 3x + 6y = 6 \end{cases}$

4. $\begin{cases} 3x + 4y = 10 \\ 7x - 5y = 9 \end{cases}$

5. $\begin{cases} .5x + .8y = .6 \\ x + 1.6y = 1.2 \end{cases}$

6. $\begin{cases} 10x - 15y = 12 \\ 14x - 21y = 16 \end{cases}$

7. $\begin{cases} 9x + 3y = 12 \\ 12x + 4y = 16 \end{cases}$

8. $\begin{cases} 2x + 5y = 11 \\ x + y = 1 \end{cases}$

120. The Multiplication-Addition Method. To solve two equations in two unknown numbers, we must find a new equation containing only one of the unknown numbers; that is, we must *eliminate* one of the numbers.

EXAMPLE. Solve $\begin{cases} 5x + 4y = 11 \\ 7x + 6y = 15 \end{cases}$

To eliminate y we look at the terms $+4y$ and $+6y$. Since 12 is the L. C. M. of 4 and 6, we should change the first equation so that it contains $+12y$, and the second equation so that it contains $-12y$. Hence we multiply the first equation by 3 and the second by -2. We shall find it convenient to show our choice of multipliers thus:

$$\begin{array}{r|l} 3 & 5x + 4y = 11 \\ -2 & 7x + 6y = 15 \end{array}$$

After multiplying both left and right members we get:

$$\begin{array}{lrcr} & 15x + 12y &=& 33 \\ & -14x - 12y &=& -30 \\ \hline \text{Add:} & x &=& 3 \end{array}$$

This work does not finish the solution because the value of y must still be found; but the pupil should notice *how* y is eliminated.

MODEL SOLUTION. Solve $\begin{cases} 5x + 3y = 14 \\ 4x + 7y = 2 \end{cases}$

First Part

$$\begin{array}{r|l} 7 & 5x + 3y = 14 \\ -3 & 4x + 7y = 2 \end{array}$$

$$\begin{array}{rcr} 35x + 21y &=& 98 \\ -12x - 21y &=& -6 \\ \hline 23x &=& 92 \\ x &=& 4 \end{array}$$

Second Part

Substitute $x = 4$ in the equation

$$5x + 3y = 14$$

Then $20 + 3y = 14$

$$3y = -6$$

$$y = -2$$

Check. Substitute $x = 4$, $y = -2$ in both equations:

$$5x + 3y = 14 \qquad 4x + 7y = 2$$
$$20 - 6 = 14 \qquad 16 - 14 = 2$$
$$14 = 14 \qquad 2 = 2$$

121. EXERCISES — PROBLEMS

Solve and check the following sets of equations:

1. $\begin{cases} 5x - 2y = 4 \\ 2x + 3y = 13 \end{cases}$

2. $\begin{cases} 4x + 5y = 7 \\ 3x + 4y = 5 \end{cases}$

3. $\begin{cases} 3x - 7y = 1 \\ 2x - 9y = 5 \end{cases}$

4. $\begin{cases} 5x + 6y = 5 \\ 2x - 5y = 2 \end{cases}$

5. $\begin{cases} 5u + 6v = 3 \\ 3u - 4v = -2 \end{cases}$

6. $\begin{cases} 2h + k = 2\frac{1}{2} \\ 12h - 4k = 0 \end{cases}$

7. $\begin{cases} 3r + 8s = 2 \\ 5r = 9 - 2s \end{cases}$

8. $\begin{cases} \frac{1}{2}u - \frac{1}{3}v = -1 \\ \frac{1}{3}u + \frac{1}{2}v = 8 \end{cases}$

9. $\begin{cases} \frac{1}{4}x + \frac{1}{2}y = 1 \\ \frac{1}{3}x = y + 4\frac{2}{3} \end{cases}$

10. $\begin{cases} .2x + .3y = .3 \\ .5x + y = .5 \end{cases}$

11. Some boys in a canoe go 17 mi. downstream in 2 hr., but need $4\frac{1}{4}$ hr. to return the same distance upstream. What would be the rate of the canoe in still water, and what is the rate of the current of the river? (If the rate in still water $= R$, and the rate of the current $= r$, then the rate going downstream is $R + r$, and the rate upstream is $R - r$.)

12. If a boat can go 21 mi. downstream in 3 hr., and 20 mi. upstream in 4 hr., find the rate of the boat in still water and the rate of the current of the river.

13. The rate of the current of a river is $1\frac{1}{2}$ mi. an hour. A boat can go a certain distance downstream in $1\frac{3}{4}$ hr. and return the same distance upstream in $3\frac{1}{4}$ hr. Find the rate of the boat in still water and the distance it went.

14. Solve the problem on page 105.

15. Find the constants a and b in $P = a + bW$ if a force, P, of 8 lb. will lift a weight, W, of 10 lb., and a force of 8.4 lb. will lift a weight of 12 lb.

16. If $P = a + bW$, and $P = 3$ when $W = 4$, and $P = 10$ when $W = 18$, find the value of P when $W = 20$. (First find the constants a and b in the formula.)

122. The Substitution Method. The method shown below can be used in more difficult situations than the method on page 108. (See page 176.)

EXAMPLE. Solve $\begin{cases} 4x + 3y = 5 & (1) \\ 6x - 7y = 19 & (2) \end{cases}$

First solve equation (1) for y, getting $y = \dfrac{5 - 4x}{3}$. Then copy equation (2), but in place of y write the fraction $\dfrac{5 - 4x}{3}$. This is permissible because y is equal to this fraction. The result is

$$6x - 7\frac{(5 - 4x)}{3} = 19$$

This equation contains only *one* unknown number, x. The unknown number y has been eliminated.

The following complete solution of another set of equations is given here as a model of how to arrange and explain the work:

MODEL SOLUTION. Solve $\begin{cases} 2x + 3y = 5 & (1) \\ 4x - 5y = 21 & (2) \end{cases}$

From (1) get $3y = 5 - 2x$, or $y = \dfrac{5 - 2x}{3}$ (3)

Use this value of y in (2): $4x - 5\dfrac{(5 - 2x)}{3} = 21$

Multiply by 3: $12x - 5(5 - 2x) = 63$

Remove parenthesis: $12x - 25 + 10x = 63$

Solve for x: $x = 4$

Substitute 4 for x in equation (3): $y = \dfrac{5 - 8}{3} = -1$

Check. Check the solution by the method on page 108.

This is called the *substitution method* because one of the equations is solved for one of the unknown numbers and then that solution is substituted in the other equation. It is not necessary to begin by solving the *first* equation for y; we may solve *either* equation for either x or y, and then substitute in the other equation.

123. EXERCISES — PROBLEMS

By the substitution method, solve:

1. $\begin{cases} 5x + 2y = 4 \\ 7x - 3y = 23 \end{cases}$

2. $\begin{cases} 5x + y = 7 \\ x - 2y = 8 \end{cases}$

3. $\begin{cases} a + b = 1 \\ 5a - 4b = 23 \end{cases}$

4. $\begin{cases} 2r + s = 7 \\ r + 2s = 8 \end{cases}$

5. $\begin{cases} 8a + 2b = 5 \\ 2a - b = -1 \end{cases}$

6. $\begin{cases} 7r + s = 5\frac{3}{7} \\ r + 3s = 2 \end{cases}$

7. $\begin{cases} 3x + 4y = 18 \\ 5x - 3y = 1 \end{cases}$

8. $\begin{cases} a - b = 0 \\ 2a + 3b = 30 \end{cases}$

9. $\begin{cases} r - 6s = 0 \\ 3r + s = 19 \end{cases}$

10. $\begin{cases} 5h + 3k = 8.4 \\ h - k = .4 \end{cases}$

11. $\begin{cases} .2r + s = 1.12 \\ r + .03s = .63 \end{cases}$

12. $\begin{cases} x + y = 4000 \\ .06x + .05y = 230 \end{cases}$

13. If a baseball team plays 3 more games and wins all, it will have won $\frac{3}{4}$ of the total games played. If it loses the 3 games, it will have won $\frac{5}{8}$ of the games played. How many games has the team won and lost so far?

14. Two weights balance on a lever when one is 10 in. and the other is 7 in. from the fulcrum. If 5 lb. is added to each weight, they balance when respectively 15 in. and 12 in. from the fulcrum. Find each weight.

15. Gold loses $\frac{4}{77}$ of its weight when weighed in water, and silver loses $\frac{2}{21}$ of its weight similarly. The crown of Hiero of Syracuse, which was part gold and part silver, weighed 20 lb. and lost $1\frac{1}{4}$ lb. when weighed in water. How much gold and how much silver did it contain?

16. When weighed in water, tin loses about $\frac{1}{7}$ of its weight and copper about $\frac{1}{9}$ of its weight. An alloy of the metals, weighing 46 lb., lost 6 lb. when weighed in water. How many pounds of each metal were there in the alloy?

124. Certain sets of equations in which x and y occur in the denominators can be solved by the multiplication-addition method without first clearing of fractions.

Thus, for the set of equations shown at the right, we multiply the first equation by 2 and the second equation by 3. The fractions in the left members of the new equations can easily be added, as they have the same denominators.

$$\frac{5}{x} - \frac{6}{y} = -\frac{4}{3}$$
$$\frac{7}{x} + \frac{4}{y} = \frac{13}{3}$$

The fraction $\frac{1}{x}$ is called the *reciprocal* of x.

125. EXERCISES

Solve the following sets by the method shown above:

1. $\begin{cases} \frac{7}{x} - \frac{5}{y} = -9 \\ \frac{2}{x} + \frac{4}{y} = 11 \end{cases}$

2. $\begin{cases} \frac{3}{x} + \frac{2}{y} = 12 \\ \frac{4}{x} - \frac{5}{y} = -7 \end{cases}$

3. $\begin{cases} \frac{3}{x} + \frac{5}{y} = 1 \\ \frac{5}{x} - \frac{3}{y} = \frac{8}{15} \end{cases}$

4. $\begin{cases} \frac{4}{u} + \frac{5}{v} = 22 \\ \frac{1}{u} + \frac{2}{v} = 8\frac{1}{2} \end{cases}$

5. $\begin{cases} \frac{1}{r} + \frac{4}{s} = 2 \\ \frac{3}{r} - \frac{8}{s} = 11 \end{cases}$

6. $\begin{cases} \frac{2}{x} + \frac{1}{3y} = 5 \\ \frac{6}{x} - \frac{5}{3y} = 7 \end{cases}$

7. $\begin{cases} \frac{9}{x} - \frac{5}{2y} = 1 \\ \frac{11}{x} - \frac{3}{2y} = 2 \end{cases}$

8. $\begin{cases} r + \frac{2}{s} = 5 \\ 2r - \frac{3}{s} = 3 \end{cases}$

9. $\begin{cases} 5r + \frac{4}{s} = 6 \\ 7r + \frac{3}{s} = 11 \end{cases}$

10. $\begin{cases} \frac{5}{u} - 3v = 7 \\ \frac{1}{u} - 2v = 3\frac{1}{2} \end{cases}$

126. Sets of Literal Equations. In the example below, the multiplication-addition method is used. The substitution method can also be used.

EXAMPLE. Solve for x and y: $\begin{cases} ax + by = 2a^2 + b^2 & (1) \\ bx - 3ay = -ab & (2) \end{cases}$

$$\begin{array}{lrcrcl}
\text{Multiply (1) by } b: & abx & + & b^2y & = & 2a^2b + b^3 \\
\text{Multiply (2) by } -a: & -abx & + & 3a^2y & = & a^2b \\ \hline
\text{Add:} & & & 3a^2y + b^2y & = & 3a^2b + b^3 \\
\text{or,} & & & y(3a^2 + b^2) & = & 3a^2b + b^3
\end{array}$$

$$\text{Hence } y = \frac{3a^2b + b^3}{3a^2 + b^2} = \frac{b(3a^2 + b^2)}{3a^2 + b^2} = b$$

You can substitute $y = b$ in either (1) or (2) and thus find the value of x. In some sets, however, it is easier to find the other unknown quantity by beginning again with equations (1) and (2), selecting two new multipliers, adding, etc.

127. EXERCISES

Solve for x and y. Use above method in ex. 1 to 4, substitution method in ex. 5 to 8, and either method in ex. 9 to 11.

1. $\begin{cases} ax + by = 2a^2b \\ x + y = a^2 + b^2 \end{cases}$

2. $\begin{cases} ax + by = 2ab \\ bx + ay = a^2 + b^2 \end{cases}$

3. $\begin{cases} 3bx - 2ay = 4a \\ 6bx + y = 4a - 1 \end{cases}$

4. $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$

5. $\begin{cases} (a - b)x - y = 0 \\ a^2x - (a + b)y = b^3 \end{cases}$

6. $\begin{cases} x + y = 2a \\ bx - by = 2 \end{cases}$

7. $\begin{cases} x + y = a + 2b \\ a(b + y) - 2bx = ab \end{cases}$

8. $\begin{cases} (a + 1)x - y = b \\ ax - (a + 1)y = -a^2b \end{cases}$

9. $\begin{cases} y + bx = ax \\ (a + b)x - y = 2ab + 2b^2 \end{cases}$

10. $\begin{cases} (a + 1)x + (a - 1)y = 2a \\ (a^2 - 1)x - (a^2 - 1)y = -4a \end{cases}$

11. $\begin{cases} (a + b)x - (a - b)y = 4ab \\ (a - b)x + (a + b)y = 2a^2 - 2b^2 \end{cases}$

SETS OF THREE LINEAR EQUATIONS

128. The Multiplication-Addition Method.

EXAMPLE. Solve for x, y, and z the set:

$$\begin{cases} 2x - 3y - z = -4 & (1) \\ 3x + y + 2z = 7 & (2) \\ 4x - 2y + 3z = -1 & (3) \end{cases}$$

The plan is to eliminate x from the first two equations, and from the last two equations. We shall thereby obtain a new set of two equations in two unknown numbers, y and z, which can be found by any of the previous methods.

Multiply (1) by -3:	$-6x + 9y + 3z = 12$	
Multiply (2) by 2:	$6x + 2y + 4z = 14$	
Add:	$11y + 7z = 26$	(4)
Multiply (2) by 4:	$12x + 4y + 8z = 28$	
Multiply (3) by -3:	$-12x + 6y - 9z = 3$	
Add:	$10y - z = 31$	(5)

The problem has thus been reduced to the solving of the two equations (4) and (5) containing only y and z:

$$11y + 7z = 26 \quad (4)$$
$$10y - z = 31 \quad (5)$$

These equations are next solved by any method. We find that $y = 3$, $z = -1$; and if we substitute these values in equation (1) we find that $x = 2$.

The solution should then be checked by substituting the values $x = 2$, $y = 3$, $z = -1$ in equations (1), (2), and (3).

Because of the many possible ways of solving sets of three equations, the pupil will find it worth while to examine all the equations carefully before selecting a method.

In the example above, what multipliers should be used to eliminate y from equations (1) and (2), and to eliminate y from equations (2) and (3)?

What multipliers should be used to eliminate z from equations (1) and (2), and from equations (1) and (3)?

129. EXERCISES — PROBLEMS

Solve the following sets for x, y, and z:

1. $\begin{cases} x + 2y - z = 8 \\ 2x - y + z = -3 \\ 3x - 2y - 2z = 5 \end{cases}$

2. $\begin{cases} 2x + 3y - 4z = -1 \\ x - 6y + 2z = 3 \\ 4x - 3y + 8z = 5 \end{cases}$

3. $\begin{cases} 5x + 2y - z = 6 \\ 7x - 3y + 2z = -10 \\ 3x - y - 3z = 4 \end{cases}$

4. $\begin{cases} x + y + z = 1 \\ x - y + z = -5 \\ x + y - z = 9 \end{cases}$

5. $\begin{cases} 5x + y - z = 5 \\ x - 3y + z = 3 \\ 2x + 3y - z = 0 \end{cases}$

6. $\begin{cases} 6x + 4y + 3z = 1 \\ 5x + y - z = 2 \\ 2x - y - 5z = 3 \end{cases}$

7. $\begin{cases} x + .2y - .3z = 1 \\ .3x - y + .2z = 8.5 \\ x + .8y + z = 11 \end{cases}$

8. $\begin{cases} x + 3y = 7 \\ 2y - 3z = 5 \\ 3x + 4y + 6z = 9 \end{cases}$

9. $\begin{cases} x + 2y = 8a \\ y + 2z = -5a \\ 2x - y = a \end{cases}$

10. $\begin{cases} x + 2y - z = 2b \\ x + z = 0 \\ 3y + z = 5b \end{cases}$

11. $\begin{cases} \frac{2}{x} + \frac{3}{y} + \frac{4}{z} = 3 \\ \frac{1}{x} + \frac{6}{y} - \frac{2}{z} = 2 \\ \frac{3}{x} + \frac{4}{y} - \frac{8}{z} = \frac{5}{6} \end{cases}$

12. $\begin{cases} \frac{3}{x} + \frac{2}{y} + \frac{1}{z} = \frac{5}{2} \\ \frac{1}{x} - \frac{1}{y} + \frac{2}{z} = -\frac{5}{2} \\ \frac{2}{x} + \frac{3}{y} - \frac{2}{z} = 6 \end{cases}$

13. The sum of three numbers is 15. (Call the numbers f, s, and t.) The sum of the first and the second is 5 more than the third number. The sum of the first and the third is 4 times the second number. Find the three numbers.

14. In a track meet between three teams, A, B, and C, the sum of their scores was 116. A and B together made 66 more points than C. B and C together made 6 more points than A. Which team won the track meet?

15. Find three numbers such that when added two at a time, the sums are 19, 6, and 7.

130. The substitution method also can be used to solve sets of three equations. The method consists in solving any one of the equations for x or y or z, and substituting the solution in the other equations.

$$\begin{aligned} 3x + 2y - 4z &= 3 \quad (1)\\ 2x + y + 3z &= 8 \quad (2)\\ 5x + 3y + 2z &= 14 \quad (3) \end{aligned}$$

For example, in the above set, solve equation (2) for y since its coefficient is 1. Substituting $8 - 2x - 3z$ for y in equations (1) and (3) reduces these equations to

$x + 10z = 13$ and $x + 7z = 10$ from which $x = 3$, $z = 1$. Then $y = 8 - 2x - 3z = 8 - 6 - 3 = -1$.

131. EXERCISES — PROBLEMS

Solve the following sets of equations by any method:

1. $\begin{cases} 2x + y - 3z = 0 \\ 3x + 2y + 2z = 7 \\ 2x + 4y - 5z = 1 \end{cases}$

2. $\begin{cases} x + 3y + 2z = 2 \\ 3x - 6y + 6z = 1 \\ 2x + 3y - 4z = 1 \end{cases}$

3. $\begin{cases} x - y - z = 0 \\ 3x - 2y + 2z = 7 \\ 2x + 3y - 8z = 4 \end{cases}$

4. $\begin{cases} \dfrac{x}{6} - \dfrac{y}{3} + \dfrac{z}{4} = 4 \\[2mm] \dfrac{x}{8} - \dfrac{y}{6} + \dfrac{z}{2} = 10 \\[2mm] \dfrac{x}{2} + \dfrac{y}{3} - \dfrac{z}{5} = 5 \end{cases}$

5. $\begin{cases} \dfrac{x}{9} + \dfrac{y}{3} - \dfrac{z}{6} = 7 \\[2mm] \dfrac{x}{2} - \dfrac{y}{4} + \dfrac{z}{3} = 4 \\[2mm] \dfrac{x}{3} - \dfrac{y}{2} - \dfrac{z}{3} = 2 \end{cases}$

6. $\begin{cases} x + 3y + 2z = 5 \\ 2x - 4y + 3z = 26 \\ 5x + 2y - 2z = 3 \end{cases}$

7. $\begin{cases} 2x + 3y - z = 0 \\ 4x + 2y - 3z = 8 \\ 3x + 2y + 4z = 5 \end{cases}$

8. $\begin{cases} 3x + 2y - z = 5 \\ 2x + 5y - 2z = 3 \\ 4x - 3y + 5z = 20 \end{cases}$

9. $\begin{cases} \dfrac{3}{x} - \dfrac{1}{y} + \dfrac{6}{z} = 1 \\[2mm] \dfrac{1}{x} + \dfrac{2}{y} - \dfrac{3}{z} = 8 \\[2mm] \dfrac{2}{x} + \dfrac{3}{y} - \dfrac{9}{z} = 11 \end{cases}$

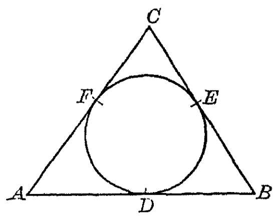
10. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} - \dfrac{1}{z} = 1\frac{1}{2} \\[2mm] \dfrac{2}{x} + \dfrac{3}{y} + \dfrac{1}{z} = 4 \\[2mm] \dfrac{5}{x} - \dfrac{8}{y} - \dfrac{1}{z} = \frac{1}{2} \end{cases}$

11. If A and B can do certain work in $6\frac{2}{3}$ days, B and C in 6 days, and A and C in $5\frac{5}{11}$ days, how many days will it take each man to do the work alone?

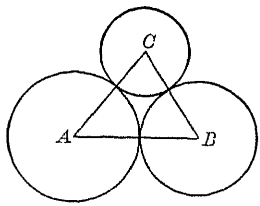
12. A train starts from station A for station B. If it goes 5 mi. an hour faster than its usual rate, it will arrive 40 min. ahead of time. If it goes 8 mi. an hour faster than its usual rate, it will arrive 1 hr. ahead of time. Find its usual rate, r, its usual time, t, and the distance, d, between the stations.

13. It is proved in geometry that $AD = AF$, $BD = BE$, etc., if the lines are tangents to the circle. Find the lengths of AD, DB, and FC if $AB = 15$ in., $BC = 13$ in., and $CA = 14$ in.

(Let $AD = x$, $DB = y$, $FC = z$.)



14. In the figure, the circles are all tangent to each other and hence AB, BC, and CA are straight lines. Find the radii of the circles if $AB = 13$ in., $BC = 11$ in., and $CA = 12$ in.



15. If the length of a rectangle is increased 1 ft. and the width is increased 2 ft., the area is increased 60 sq. ft. If, however, the length is decreased 2 ft. and the width is decreased 1 ft., the area is decreased 54 sq. ft. Find the length and the width of the rectangle.

16. Find the equations in the following problem and tell how they differ from the previous ones:

If a rectangle were 1 ft. longer and 2 ft. wider, its area would be 380 sq. ft. If the rectangle were 2 ft. less in length, and 3 ft. wider, its area would be 340 sq. ft. Find the length and the width of the rectangle.

Such sets of equations are studied in Chapter XII.

132. MISCELLANEOUS EXERCISES

1. By eliminating t from the two equations
$$v = gt \quad \text{and} \quad s = \tfrac{1}{2}gt^2$$
find an equation that does not contain t. (Solve one of the equations for t, and substitute in the other.)

2. By eliminating t from $x = t + 3$ and $y = 2t + 5$, express y in terms of x. (This means: find an expression for y that does not contain t but does contain x.)

3. By eliminating t from $x = t - 2$ and $y = t^2 + 3t + 1$, express y as a function of x. (See the method in ex. 2.)

4. Eliminate r from the two equations
$$x = r + 1 \quad \text{and} \quad y = 3r^2 + 2$$

5. Eliminate r from the equations $C = 2\pi r$ and $A = \pi r^2$. (The result is a formula for A in terms of C.)

6. Eliminate t from: $x = t^2 + 1$, $y = t^2 - 1$

7. Eliminate t from: $x = t - 1$, $y = t^3 - 1$

8. If $z = x^2 - y^2$ and $x = 3t$, $y = 2t + 5$, express z as a function of t.

9. If $h = t(100 + d)$ and also $h = 100T$, find an expression for d in terms of t and T.

10. If $y = 1 - s^2 + s^4$ and $x = 1 - s^2$, express y as a function of x.

11. A rod made of lead expands when heated. If l is its length in inches at a temperature of t degrees centigrade, then $l = a + bt$ where a and b are certain constants.

In an experiment it was found that
$l = 100.00540$ when $t = 20$, and $l = 100.02430$ when $t = 90$.
Find the constants a and b to five decimal places.

12. In an experiment (like that in ex. 11) with a copper rod, it was found that
$l = 40.00390$ when $t = 60$, and $l = 40.00520$ when $t = 80$.
Find the constants a and b to five decimal places.

133. Representing a Number by Its Digits. A number like 548 is a three-digit number; 26 is a two-digit number.

If the digits of a two-digit number are t and u, how shall we represent the number? We cannot write $t + u$, for that is the sum of the digits. Neither can we write tu because that means the product of the digits.

We can find the correct expression if we notice that 26 is our way of writing $20 + 6$, or $10 \cdot 2 + 6$. In the number 548, for example, the digit 4 means 40, or $10 \cdot 4$, and the digit 5 means really 500, or $100 \cdot 5$.

A two-digit number is therefore represented by $10\,t + u$. How is a three-digit number represented? How can you represent the number formed when the digits are reversed?

By expressing a number in terms of its digits, many interesting facts about numbers can be proved.

134. EXERCISES

1. Subtracting 28 from 82 gives 54. Subtracting 37 from 73 gives 36. Show algebraically that if any number is subtracted from the number formed by reversing the digits, the result is exactly divisible by 9.

2. A three-digit number can be written $100\,h + 10\,t + u$, which can be written as $9(11\,h + t) + (h + t + u)$. From this expression prove the rule:

If the sum of the digits of a number is divisible by 9, *then the number itself is divisible by* 9.

3. If a number ends in 5, it can be written $10\,t + 5$. Find $(10\,t + 5)^2$ and show that this square can be written as $100\,t(t + 1) + 25$. Notice that t and $(t + 1)$ are consecutive integers. Now explain the rule:

To square a number ending in 5, *multiply the tens' digit by the next higher digit, and then write* 25 *at the end.* Thus:

$$75^2 = 5625 \text{ (because } 7 \cdot 8 = 56)$$

(Supplementary Topic, Pages 120, 121)

135. Determinants. If we solve for x and y, the set

$$ax + by = c$$
$$dx + ey = f$$

we find that $x = \dfrac{ce - bf}{ae - bd}$, $y = \dfrac{af - cd}{ae - bd}$. These formulas can be written in a more useful form.

First, we agree that an expression like $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ consisting of four numbers written in two columns, with the vertical lines at their sides, shall mean:

the product ***ae*** *minus* *the product* ***bd.***

Such an arrangement of numbers is called a *determinant.*

From the six numbers $\begin{matrix} a & b & c \\ d & e & f \end{matrix}$ that appear in the equations, we can form three useful determinants:

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}, \quad \begin{vmatrix} a & c \\ d & f \end{vmatrix}, \quad \begin{vmatrix} c & b \\ f & e \end{vmatrix}$$

The formulas for x and y can then be written:

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

These formulas can easily be remembered by noticing that

1. The denominators in the formulas are the same and consist of the coefficients of x and y.

2. The numerator in the formula for x is found by substituting the constants, c and f, in the right members for the coefficients, a and d, of x.

3. The numerator in the formula for y is found by substituting the constants, c and f, in the right members for the coefficients, b and e, of y.

EXAMPLE. By determinants, solve $\begin{cases} 5x + 9y = 44 \\ 11x - 7y = 3 \end{cases}$

$$x = \frac{\begin{vmatrix} 44 & 9 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 5 & 9 \\ 11 & -7 \end{vmatrix}} = \frac{44(-7) - 3(9)}{5(-7) - 11(9)} = \frac{-308 - 27}{-35 - 99} = \frac{-335}{-134} = \frac{5}{2}$$

Since the denominator for y is the same as that for x,

$$y = \frac{\begin{vmatrix} 5 & 44 \\ 11 & 3 \end{vmatrix}}{-134} = \frac{5(3) - 11(44)}{-134} = \frac{15 - 484}{-134} = \frac{-469}{-134} = \frac{7}{2}$$

The pupil should solve this set by some other method and notice that he must perform the same multiplications; namely, multiply 44 by -7, -3 by 9, 5 by -7, etc. This will always be so when the coefficients, a, b, d, and e, are prime to each other. Determinants, however, enable us to work rapidly even though the arithmetic work is the same.

136. EXERCISES

By determinants, solve:

1. $\begin{cases} 12x + 5y = 11 \\ 7x + 3y = 6 \end{cases}$

2. $\begin{cases} 6x + 9y = 5 \\ 5x + 8y = 4 \end{cases}$

3. $\begin{cases} 6x - 5y = 7 \\ 7x - 6y = 9 \end{cases}$

4. $\begin{cases} 5x + 7y = 34 \\ 2x + 3y = 16 \end{cases}$

5. $\begin{cases} x - .1y = 1.7 \\ .5x + y = 4 \end{cases}$

6. $\begin{cases} 3x - 7y = 100 \\ .2x + .05y = 110 \end{cases}$

7. $\begin{cases} 1.2x - 1.7y = 8 \\ .4x + 1.1y = 5 \end{cases}$

8. $\begin{cases} 4x - .1y = 30 \\ .2x + .03y = 5 \end{cases}$

9. $\begin{cases} (a + b)x - (a - b)y = 6ab \\ (a - b)x + (a + b)y = 2a^2 - 4b^2 \end{cases}$

CHAPTER VIII

RADICALS

137. Review. A *radical* is an indicated root of any algebraic or arithmetic expression, as $\sqrt{9}$, $\sqrt[3]{5}$, $\sqrt{ab}$.

The small figure, like the ^{3} in $\sqrt[3]{5}$, which indicates what root is required, is called the *index* of the radical. When no index is given, as in $\sqrt{5}$, the index ^{2} is understood.

The *radicand* is the number or expression whose root is required, as $3\,ay$ in $\sqrt{3\,ay}$, or $a+b$ in $\sqrt{a+b}$.

Although the square root of 9 is both $+3$ and -3, the symbol $\sqrt{9}$ means only $+3$, and $-\sqrt{9}$ means only -3.

In earlier work we have learned the following rules:

RULE 1. $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$ Thus, $\sqrt{9}\cdot\sqrt{16}=\sqrt{144}$

RULE 2. $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$ Thus, $\dfrac{\sqrt{9}}{\sqrt{16}}=\sqrt{\dfrac{9}{16}}$

But $\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$ Thus, $\sqrt{9}+\sqrt{4}\neq\sqrt{13}$

$\sqrt{a}-\sqrt{b}\neq\sqrt{a-b}$ Thus, $\sqrt{9}-\sqrt{4}\neq\sqrt{5}$

These relations are equally important when written as:

RULE 1′. $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$ Thus, $\sqrt{100}=\sqrt{25}\cdot\sqrt{4}$

RULE 2′. $\sqrt{\dfrac{a}{b}}=\dfrac{\sqrt{a}}{\sqrt{b}}$ Thus, $\sqrt{\dfrac{4}{9}}=\dfrac{\sqrt{4}}{\sqrt{9}}$

But $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$ Thus, $\sqrt{34}\neq\sqrt{25}+\sqrt{9}$

$\sqrt{a-b}\neq\sqrt{a}-\sqrt{b}$ Thus, $\sqrt{60}\neq\sqrt{64}-\sqrt{4}$

These relations have been stated here only for square roots but they are also correct for any other root.

138. ORAL EXERCISES

What rule is illustrated by each of the following?

1. $\sqrt{2}\sqrt{32}=\sqrt{64}$
2. $\sqrt{50}=\sqrt{25}\sqrt{2}$
3. $\sqrt{x}\sqrt{xy^2}=\sqrt{x^2y^2}$
4. $\sqrt[3]{120}=\sqrt[3]{8}\sqrt[3]{15}$

Using Rule 1′, state in some other form the products:

5. $\sqrt{2}\sqrt{8}$ $\quad -\sqrt{2}\sqrt{8}$ $\quad \sqrt{3}\sqrt{27}$ $\quad \sqrt{a}\sqrt{a^3}$
6. $\sqrt[3]{4}\sqrt[3]{2}$ $\quad -\sqrt[3]{4}\sqrt[3]{16}$ $\quad \sqrt[3]{12}\sqrt[3]{5\frac{1}{3}}$ $\quad \sqrt[3]{a^2}\sqrt[3]{a^4}$
7. $\sqrt{\frac{1}{2}}\sqrt{\frac{9}{8}}$ $\quad \sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}}$ $\quad -\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}$ $\quad -\sqrt{\frac{1}{5}}\sqrt{\frac{4}{5}}$
8. $\sqrt[3]{\frac{1}{2}}\sqrt[3]{\frac{1}{4}}$ $\quad \sqrt[3]{\frac{2}{3}}\sqrt[3]{\frac{4}{9}}$ $\quad \sqrt[3]{\frac{3}{4}}\sqrt[3]{\frac{9}{16}}$ $\quad \sqrt[3]{\frac{4}{9}}\sqrt[3]{\frac{16}{3}}$

Using Rule 1, tell how each of the following radicals can be written as the product of two other radicals:

9. $\sqrt{8}$ $\quad \sqrt{50}$ $\quad \sqrt{75}$ $\quad \sqrt{60}$ $\quad \sqrt{24}$ $\quad \sqrt{32}$
10. $\sqrt[3]{16}$ $\quad \sqrt[3]{81}$ $\quad \sqrt[3]{24}$ $\quad \sqrt[3]{40}$ $\quad \sqrt[3]{125}$ $\quad \sqrt[3]{250}$

Using Rule 2 or Rule 2′, state in some other form:

11. $\dfrac{\sqrt{15}}{\sqrt{3}}$ $\quad \sqrt{\dfrac{a^2}{a}}$ $\quad \sqrt{\dfrac{bc}{b}}$ $\quad \sqrt{\dfrac{x^2-y^2}{x-y}}$ $\quad \sqrt{\dfrac{a^3+b^3}{a+b}}$

Which of the following equations have been solved correctly? State the errors in the other solutions.

12. $2x^2=25$, $\quad 2x=\pm 5$, $\quad x=\pm 2\frac{1}{2}$
13. $4x^2=100$, $\quad 2x=10$, $\quad x=5$
14. $x^2+a^2=b^2$, $\quad x+a=b$, $\quad x=b-a$

Which of the following statements are correct? State which rule the incorrect statements violate:

15. $\frac{1}{2}\sqrt{7}+\frac{1}{3}\sqrt{7}=\frac{5}{6}(2\sqrt{7})$
16. $\frac{3}{5}-\frac{2}{5}\sqrt{6}=\frac{1}{5}\sqrt{6}$
17. $\frac{1}{5}\sqrt{6}\cdot\frac{1}{2}\sqrt{6}=\frac{1}{10}\sqrt{6}$
18. $\sqrt{4r^2+r^2}=2r+r$
19. $\sqrt{\frac{2}{3}}(\sqrt{\frac{3}{2}}+1)=1+\sqrt{\frac{2}{3}}$
20. $\sqrt{\frac{5}{7}}(\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{6}})=\sqrt{\frac{5}{7}}\sqrt{\frac{1}{3}}$
21. $\sqrt{\frac{3}{7}}(\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}})=\sqrt{\frac{3}{7}}\sqrt{\frac{5}{6}}$
22. $\sqrt{4r^2-r^2}=r\sqrt{3}$

139. Changes in the Radicand. When working with radicals it is often desirable to change the radicand so that

a. The numerical coefficient and the degree of the radicand are as small as possible.

b. The radicand is not a fraction.

In making these changes we use the rules on page 122.

EXAMPLES

Reducing the numerical coefficient and the degree:

1. $\sqrt{50\,a^3} = \sqrt{25\,a^2}\sqrt{2\,a} = 5\,a\sqrt{2\,a}$
2. $\sqrt{(x+y)^3} = \sqrt{(x+y)^2}\sqrt{(x+y)} = (x+y)\sqrt{x+y}$
3. $\sqrt[3]{32\,r^4} = \sqrt[3]{8\,r^3}\sqrt[3]{4\,r} = 2\,r\sqrt[3]{4\,r}$
4. $\sqrt[3]{x^5y} = \sqrt[3]{x^3}\sqrt[3]{x^2y} = x\sqrt[3]{x^2y}$
5. $\sqrt[4]{x^5y} = \sqrt[4]{x^4}\sqrt[4]{xy} = x\sqrt[4]{xy}$

Changing a fractional radicand to an integral expression:

6. $\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \sqrt{\frac{1}{16}}\sqrt{10} = \frac{1}{4}\sqrt{10}$
7. $\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{\frac{1}{b^2}}\sqrt{ab} = \frac{1}{b}\sqrt{ab}$
8. $\sqrt{\frac{x+y}{x-y}} = \sqrt{\frac{(x+y)(x-y)}{(x-y)^2}} = \frac{1}{x-y}\sqrt{x^2-y^2}$
9. $\sqrt[3]{\frac{a}{b}} = \sqrt[3]{\frac{ab^2}{b^3}} = \sqrt[3]{\frac{1}{b^3}}\sqrt[3]{ab^2} = \frac{1}{b}\sqrt[3]{ab^2}$
10. $\sqrt[3]{\frac{a}{b^5}} = \sqrt[3]{\frac{ab}{b^6}} = \sqrt[3]{\frac{1}{b^6}}\sqrt[3]{ab} = \frac{1}{b^2}\sqrt[3]{ab}$

When changing a radicand, use the following processes:

RULES. *Factor the radicand, choosing factors at least one of which is a square or a cube, etc., so that you can find its square root or cube root, etc.*

If the radicand is a fraction, multiply the numerator and the denominator by some quantity so that you can find the square root or the cube root, etc., of the denominator, and then factor.

140. EXERCISES

Reduce the following expressions, as directed in § 139:

1. $\sqrt{12a^2}$ $\quad\sqrt{18a^2}$ $\quad\sqrt{18a^7}$ $\quad\sqrt{50x^5}$ $\quad\sqrt{50b^9}$

2. $\sqrt{16a^2b^3}$ $\quad\sqrt{32a^3b^2}$ $\quad\sqrt{50a^4b^3}$ $\quad\sqrt{75x^4y}$ $\quad\sqrt{75xy^5}$

3. $\sqrt{\frac{c}{d}}$ $\quad\sqrt{\frac{2c}{d}}$ $\quad\sqrt{\frac{c}{2d}}$ $\quad\sqrt{\frac{2x}{3y}}$ $\quad\sqrt{\frac{3x}{4y}}$

4. $\sqrt{\frac{x^2}{y}}$ $\quad\sqrt{\frac{x}{y^2}}$ $\quad\sqrt{\frac{x}{2y}}$ $\quad\sqrt{\frac{x^3}{y}}$ $\quad\sqrt{\frac{x}{y^3}}$

5. $\sqrt[3]{a^4}$ $\quad\sqrt[3]{-b^5}$ $\quad\sqrt[3]{-a^3b^5}$ $\quad\sqrt[3]{a^3b^2}$

6. $\sqrt[3]{54a}$ $\quad\sqrt[3]{32a^3b^7}$ $\quad\sqrt[3]{16x^4y}$ $\quad\sqrt[3]{-108ab^4}$

7. $\sqrt[3]{\frac{x}{y}}$ $\quad\sqrt[3]{\frac{2x}{y}}$ $\quad\sqrt[3]{-\frac{3x}{y^2}}$ $\quad\sqrt[3]{-\frac{x^2}{3y}}$

8. $\sqrt[3]{\frac{2a}{3c}}$ $\quad\sqrt[3]{\frac{8a^2}{3c}}$ $\quad\sqrt[3]{\frac{16r^2}{3s}}$ $\quad\sqrt[3]{\frac{54r^2}{125s}}$

9. $\sqrt[4]{8a^4}$ $\quad\sqrt[4]{32a^5}$ $\quad\sqrt[4]{625x^9}$ $\quad\sqrt[4]{81y^7}$

10. $\sqrt{\frac{a+b}{a-b}}$

11. $\sqrt{1+\frac{a}{b}}$

12. $\sqrt{r^2+\left(\frac{r}{2}\right)^2}$

13. $\sqrt{r^2-\left(\frac{r}{2}\right)^2}$

14. $\sqrt[3]{x^3+\left(\frac{x}{2}\right)^3}$

15. $\sqrt[3]{\frac{r^2}{4}+\frac{r^2}{8}}$

16. $\sqrt{\frac{x^2+y^2}{x+y}}$

17. $\sqrt{\frac{a-b}{a^2+2ab+b^2}}$

18. $\sqrt{\frac{a^2+2a+1}{a-1}}$

19. $\sqrt{\frac{a^2+b^2}{ab}+2}$

20. $\sqrt{\left(\frac{x-1}{2}\right)^2+x}$

21. $\sqrt{\frac{x^2+y^2}{xy}-2}$

22. $\sqrt{(a+b)^2+(a-b)^2-(a^2+b^2)}$

141. Addition of Radicals. Similar radicals are those whose indices are the same and whose radicands are the same or can be made so by some change. We add similar radicals just as we add similar terms. Thus:

$$\sqrt{32}+\sqrt{50}=4\sqrt{2}+5\sqrt{2}=9\sqrt{2}$$

$$\sqrt[3]{8a^3b}-5\sqrt[3]{b^4}=2a\sqrt[3]{b}-5b\sqrt[3]{b}=(2a-5b)\sqrt[3]{b}$$

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{1}{y}\sqrt{xy}+\frac{1}{x}\sqrt{xy}=\left(\frac{1}{y}+\frac{1}{x}\right)\sqrt{xy}=\frac{x+y}{xy}\sqrt{xy}$$

But $\sqrt{50}+\sqrt{48}=5\sqrt{2}+4\sqrt{3}$, which are not similar.

142. EXERCISES

Add the following terms, where possible:

1. $3\sqrt{5}+8\sqrt{5}+2\sqrt{5}$
2. $6\sqrt{3}+2\sqrt{5}+7\sqrt{3}$
3. $\sqrt{2}+\sqrt{5}-\sqrt{7}$
4. $\sqrt{12}+\sqrt{75}+\sqrt{48}$
5. $\sqrt{45}-\sqrt{20}+\sqrt{5}$
6. $\sqrt[3]{24}+\sqrt[3]{250}-\sqrt[3]{16}$
7. $\sqrt{\frac{1}{2}}+\sqrt{\frac{9}{2}}+\sqrt{50}$
8. $\sqrt{\frac{5}{8}}-\sqrt{\frac{1}{10}}+\sqrt{90}$
9. $\sqrt{\frac{1}{12}}+\sqrt{\frac{3}{4}}-\sqrt{\frac{1}{3}}$
10. $\sqrt[3]{\frac{5}{9}}-\sqrt[3]{\frac{3}{25}}+\sqrt[3]{\frac{15}{64}}$
11. $\sqrt{18x}+\sqrt{50x}$
12. $\sqrt{50a}-\sqrt{8a}$
13. $\sqrt{50a^3}-\sqrt{18a^3}$
14. $\sqrt[3]{16y}+\sqrt[3]{54y}$
15. $\sqrt[3]{54r^4}-\sqrt[3]{16r^4}$
16. $\sqrt[3]{250a^2}+\sqrt[3]{16a^2}$
17. $\sqrt[3]{\frac{1}{4}a}+\sqrt[3]{\frac{2}{27}a}$
18. $\sqrt[4]{a^5}+\sqrt[4]{a^9}$
19. $\sqrt[4]{ab^5}+\sqrt[4]{a^5b}$
20. $\sqrt[3]{-54}+\sqrt[3]{-16}$
21. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}+\sqrt{\frac{1}{xy}}$ (By experiment see whether reducing the fractions to a common denominator is of any help.)
22. $\sqrt{\frac{a^2+b^2}{ab}+2}+\sqrt{\frac{a^2+b^2}{ab}-2}$
23. $\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}+\frac{2b}{a^2-b^2}\sqrt{a^2-b^2}$

143. Multiplication of Radicals. To check the solutions of some quadratic equations (page 153), it will be necessary to multiply expressions containing radicals. The work is like the multiplication of polynomials (page 18). Thus:

1.
$$\begin{array}{l} 2\sqrt{3} + 4\sqrt{5} \\ \underline{7\sqrt{3} - \sqrt{5}} \\ 14\sqrt{9} + 28\sqrt{15} \\ \underline{\phantom{14\sqrt{9}} - 2\sqrt{15} - 4\sqrt{25}} \\ 42 \quad + 26\sqrt{15} - \quad 20, \quad \text{or} \quad 22 + 26\sqrt{15} \end{array}$$

2.
$$\begin{aligned} &(5\sqrt{2\,a} + 4\sqrt{b})(2\sqrt{3\,a} - 3\sqrt{b}) \\ &= 10\sqrt{6\,a^2} - 15\sqrt{2\,ab} + 8\sqrt{3\,ab} - 12\sqrt{b^2} \\ &= 10\,a\sqrt{6} - 15\sqrt{2\,ab} + 8\sqrt{3\,ab} - 12\,b \end{aligned}$$

The rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ is not true if both a and b are negative (see page 146). Hence we assume here that a and b are positive.

144. EXERCISES

Find the following products in the simplest form:

1. $\sqrt{3}(\sqrt{5} - \sqrt{2})$
2. $\sqrt{2}(\sqrt{3} + \sqrt{8})$
3. $\sqrt{a}(\sqrt{ab} + \sqrt{b})$
4. $\sqrt[3]{2}(\sqrt[3]{4} - \sqrt[3]{5})$
5. $\sqrt[3]{a}(\sqrt[3]{a^2} + \sqrt[3]{ab})$
6. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{6})$
7. $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$
8. $(x - \sqrt{x - 5})^2$
9. $(\sqrt{a} + \sqrt{a - 3})^2$
10. $(\sqrt{y + 1} - y)^2$

11. Show that $x = 5 + \sqrt{3}$ is a solution of the equation $x^2 - 10\,x + 22 = 0$; that is, prove that when $5 + \sqrt{3}$ is substituted for x in the equation, the expression

$$(5 + \sqrt{3})^2 - 10(5 + \sqrt{3}) + 22$$

reduces to zero.

12. Prove that $x = 2 - \sqrt{5}$ is a solution of $x^2 - 4\,x = 1$.

13. Prove that $x = 3 + \sqrt{6}$ is a solution of $x^2 + 3 = 6\,x$.

145. Division of Radicals. According to Rule 2, page 122, $\sqrt{18} \div \sqrt{6} = \sqrt{3}$. Frequently, however, we can perform the divisions more easily by first multiplying the numerator and the denominator by some quantity that will make the denominator rational.

EXAMPLES

1. $\dfrac{15}{\sqrt{3}} = \dfrac{15 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \dfrac{15\sqrt{3}}{\sqrt{9}} = \dfrac{15\sqrt{3}}{3} = 5\sqrt{3}$

2. $\dfrac{11}{\sqrt[3]{2}} = \dfrac{11 \cdot \sqrt[3]{4}}{\sqrt[3]{2} \cdot \sqrt[3]{4}} = \dfrac{11\sqrt[3]{4}}{\sqrt[3]{8}} = \dfrac{11}{2}\sqrt[3]{4}$

3. $$\frac{12}{\sqrt{5} - \sqrt{3}} = \frac{12 \cdot (\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} = \frac{12(\sqrt{5} + \sqrt{3})}{\sqrt{25} - \sqrt{9}}$$
$$= \frac{12(\sqrt{5} + \sqrt{3})}{5 - 3} = 6(\sqrt{5} + \sqrt{3})$$

4. $$\frac{x + \sqrt{y}}{a + \sqrt{b}} = \frac{(x + \sqrt{y})(a - \sqrt{b})}{(a + \sqrt{b})(a - \sqrt{b})}$$
$$= \frac{ax - x\sqrt{b} + a\sqrt{y} - \sqrt{by}}{a^2 - b}$$

Changing a fraction so that no radical appears in the denominator is called *rationalizing* the denominator.

The numerator and the denominator are multiplied by some expression to make the denominator rational. The multiplier that is used is called *the rationalizing factor*. The radicals do not disappear entirely; they vanish from the denominator while others appear in the numerator.

Pairs of quantities, like $\sqrt{a} - \sqrt{b}$ and $\sqrt{a} + \sqrt{b}$ or $a + \sqrt{b}$ and $a - \sqrt{b}$, that are the same except for the signs of one of the terms, are called *conjugate quantities*. Notice that the product of two conjugate quantities is a rational expression.

146. EXERCISES

1. How does the work of simplifying the fraction $\frac{\sqrt{15}}{\sqrt{3}}$ differ from the work of simplifying $\frac{15}{\sqrt{3}}$? Find the value of each fraction, using the table on page 271.

2. Write in another way and then find the value of:

$$\frac{8}{2\sqrt{3}} \qquad \frac{8}{\sqrt{3}} \qquad \frac{\sqrt{18}}{\sqrt{6}} \qquad \frac{18}{\sqrt{6}} \qquad \frac{12\sqrt{10}}{\sqrt{2}} \qquad \frac{10}{\sqrt[3]{2}}$$

Change the following fractions to fractions respectively equal with rational denominators:

3. $\frac{\sqrt{5}}{\sqrt{3}-\sqrt{2}}$

4. $\frac{\sqrt{7}}{\sqrt{5}+\sqrt{3}}$

5. $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

6. $\frac{2\sqrt{3}+\sqrt{7}}{\sqrt{7}-\sqrt{5}}$

7. $\frac{3\sqrt{2}+\sqrt{6}}{3-2\sqrt{2}}$

8. $\frac{x-\sqrt{3}}{x+\sqrt{5}}$

9. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

10. $\frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+3\sqrt{y}}$

11. $\frac{2\sqrt{r}-\sqrt{s}}{r+2\sqrt{s}}$

Find the indicated quotients:

12. $(\sqrt{7}+\sqrt{5})\div(\sqrt{7}-\sqrt{3})$

13. $(a-\sqrt{b})\div(a+2\sqrt{b})$

14. $(a^2-a-b)\div(a-\sqrt{a+b})$

15. Is there any difference between the directions for ex. 3 to 11 and the directions for ex. 12 to 14?

After rationalizing the denominator of each fraction, combine the following fractions as far as possible:

16. $\frac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}} - \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} + \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

17. $\frac{2}{\sqrt{3}-1} - \frac{2}{\sqrt{3}+\sqrt{2}} - \frac{2}{\sqrt{2}+1}$

18. $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} - \frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}} - \frac{a-b}{a+b}$

147. MISCELLANEOUS EXERCISES

Find the indicated powers of:

1. $(\sqrt{2})^7$ ANS. $8\sqrt{2}$
2. $(\sqrt{3})^8$
3. $(\sqrt[3]{5})^7$
4. $(\sqrt[4]{3})^9$
5. $(\frac{1}{2}\sqrt{2})^7$
6. $(-\sqrt{2})^9$
7. $(\frac{1}{3}\sqrt{2})^5$

Change the following fractions so that the denominators will be rational:

8. $\dfrac{a\sqrt{b} - c\sqrt{d}}{a\sqrt{b} + c\sqrt{d}}$

9. $\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}$

10. $\dfrac{\sqrt{a^2 - b^2} + \sqrt{a^2 + b^2}}{\sqrt{a^2 - b^2} - \sqrt{a^2 + b^2}}$

11. $\dfrac{\sqrt{x^2 - y^2} + x}{x - \sqrt{x^2 - y^2}}$

12. By means of the equations $A = s^2$ and $d = s\sqrt{2}$ prove that the area of a square is one half the square of its diagonal.

13. If $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ prove that (*a*) $r_1 r_2 = \dfrac{c}{a}$ and (*b*) $r_1 + r_2 = -\dfrac{b}{a}$.

14. Show that $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$ and $x = \dfrac{a}{b}\sqrt{b^2 - y^2}$ are different forms of the equation $b^2x^2 + a^2y^2 = a^2b^2$.

15. In a right triangle with sides 2 in. and 3 in., the length of the hypotenuse is $\sqrt{13}$ in. Hence we have here a method for constructing a line whose length is an irrational number, $\sqrt{13}$. Using this method tell how to draw lines that will represent the following numbers:

(*a*) $\sqrt{10}$ (*b*) $\sqrt{34}$ (*c*) $\sqrt{58}$ (*d*) $\sqrt{61}$

16. From the relation $\sqrt{19} = \sqrt{4^2 + (\sqrt{3})^2}$ tell how to draw a line that will represent $\sqrt{19}$.

17. Find another expression, like the one in ex. 16, for the number $\sqrt{19}$.

(Supplementary Topics, Pages 131 to 133)

148. Square Roots of $a \pm 2\sqrt{b}$. The square of a binomial is usually a trinomial, but the square of $(\sqrt{3}+\sqrt{5})$ is $3 + 2\sqrt{15} + 5$ or $8 + 2\sqrt{15}$, which is a binomial because the terms 3 and 5 can be added. Notice that:

The radicand, 15, is the product of two numbers, 3 and 5, of which the sum is the other term, 8; and

The radical, $2\sqrt{15}$, has a coefficient 2.

These ideas enable us to find, by experiment, the square roots of expressions like $a \pm 2\sqrt{b}$.

EXAMPLE 1. Find the square roots of $12 - 2\sqrt{35}$.

Since $35 = 7 \cdot 5$, and $12 = 7 + 5$, we write -

$$12 - 2\sqrt{35} = 7 - 2\sqrt{35} + 5 = (\sqrt{7} - \sqrt{5})^2$$

The square roots of $12 - 2\sqrt{35}$ are $\pm(\sqrt{7} - \sqrt{5})$

or $$\sqrt{12 - 2\sqrt{35}} = \sqrt{7} - \sqrt{5}. \quad \text{(Why?)}$$

EXAMPLE 2. Find the square roots of $14 + \sqrt{192}$.

First change $\sqrt{192}$ to $2\sqrt{48}$ (not to $8\sqrt{3}$) because the coefficient of the radical must be 2. Then

$$14 + \sqrt{192} = 14 + 2\sqrt{48} = 8 + 2\sqrt{48} + 6$$

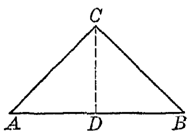
The square roots are $\pm(\sqrt{8} + \sqrt{6})$ or $\pm(2\sqrt{2} + \sqrt{6})$.

149. EXERCISES

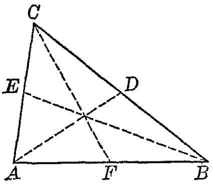
Find the square roots of the following quantities:

1. $7 + 2\sqrt{10}$
2. $10 + 2\sqrt{21}$
3. $13 - 2\sqrt{30}$
4. $8 - 2\sqrt{15}$
5. $6 - 2\sqrt{5}$
6. $14 - 2\sqrt{33}$.
7. $28 + \sqrt{300}$
8. $12 - \sqrt{80}$
9. $17 + 4\sqrt{15}$
10. $12 - 3\sqrt{12}$
11. $9 - 3\sqrt{8}$
12. $12\,a + 2\,a\sqrt{35}$
13. $2\,a + 2\sqrt{a^2 - 1}$
14. $a + \sqrt{a^2 - b^2}$
15. $2\,x + 4\,y + 2\sqrt{x^2 + 4\,xy + 3\,y^2}$

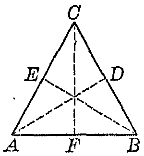
150. Definitions and Theorems from Geometry. To solve the problems on page 133 you must have in mind certain geometric relations stated below. Read them now, and later when working the problems select and read again the particular facts needed in the problem.



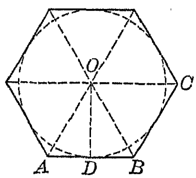
In an *isosceles* triangle the bisector of the vertex angle bisects the base and is perpendicular to the base.



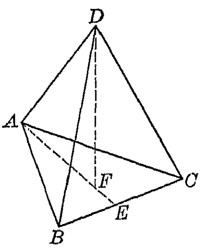
In *any* triangle the medians are concurrent (meet in a point). The point of intersection is two thirds of the distance from the vertex to the mid-point of the opposite side.



In an *equilateral* triangle the medians are also the altitudes of the triangle. They intersect at the center of the inscribed circle.



A *regular hexagon* may be divided into six equilateral triangles. The center of its inscribed circle is called the center of the hexagon. The radius of the inscribed circle is called the apothem of the hexagon.



A *regular tetrahedron* is a pyramid whose four faces are equilateral triangles. The altitude of the tetrahedron meets the base at the point where the altitudes of the base intersect.

The volume of any *pyramid* or cone is $\frac{1}{3}$ the product of the area of the base by the altitude of the pyramid or cone.

151. **GEOMETRIC EXERCISES**

Draw a figure for each problem. If the answers are expressed in radicals, give the simplest form of the radical.

1. Find the altitude and the area of an equilateral triangle whose side is a.

2. Find the area of a regular hexagon whose side is a.

3. Find the area of an isosceles triangle whose sides are $3\,s$, $3\,s$, and $2\,s$.

4. Find the radius of a circle inscribed in an equilateral triangle whose side is a.

5. Find the radius of a circle circumscribed about an equilateral triangle whose side is a.

6. The altitude of an equilateral triangle is h. Find the sides of the triangle and its area.

7. The radius of a circle inscribed in an equilateral triangle is r. Find the sides of the triangle.

8. The side of an equilateral triangle is a. Find how far the center of the inscribed circle is from any vertex and from any side.

9. If each edge of a regular tetrahedron is 20 in., find the volume of the tetrahedron.

10. Prove that the altitude of a regular tetrahedron whose edge is e is $\frac{1}{3}\,e\sqrt{6}$. Show that the volume is $\frac{1}{12}\,e^3\sqrt{2}$.

11. The base of a pyramid is a square whose sides are s. The other four edges of the pyramid are e. Find the altitude and the volume of the pyramid.

12. If s is half the perimeter of a triangle whose sides are a, b, and c, the area of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Use this formula to find the area of a triangle whose sides are 8 in., 10 in., and 12 in.

13. By means of the formula in ex. 12, find the area of an equilateral triangle whose sides are a.

CHAPTER IX

NEGATIVE AND FRACTIONAL EXPONENTS

152. By reviewing the laws for positive integral exponents, we shall see the advantages of introducing negative and fractional exponents. Just as negative numbers enable us to write several rules as a single rule, so negative and fractional exponents permit the use of fewer rules.

I. Law of Multiplication. $x^a \cdot x^b = x^{a+b}$

If a and b are positive integers then, by definition,

$$\begin{aligned} x^a &= x \cdot x \cdot x \cdot x \cdots && (a \text{ factors}) \\ x^b &= x \cdot x \cdot x \cdot x \cdots && (b \text{ factors}) \\ \text{Hence} \qquad x^a \cdot x^b &= x \cdot x \cdot x \cdot x \cdots && (a + b \text{ factors}) \\ &= x^{a+b} && \text{by definition} \end{aligned}$$

II. Law of Division. $x^a \div x^b = x^{a-b}$

If a and b are positive integers then, by definition,

$$x^a \div x^b = \frac{x^a}{x^b} = \frac{x \cdot x \cdot x \cdots (a \text{ factors})}{x \cdot x \cdot x \cdots (b \text{ factors})}$$

Here, unlike the case for multiplication, we must state whether $a>b$ or $b>a$. If $a>b$, then b factors cancel from the numerator and denominator of the above fraction and $(a - b)$ factors are left in the *numerator*. If, however, $b>a$, then a factors cancel and $(b - a)$ factors are left in the *denominator*. Hence

$$\frac{x^a}{x^b} = x^{a-b} \text{ if } a>b \qquad\qquad \frac{x^a}{x^b} = \frac{1}{x^{b-a}} \text{ if } b>a$$

We shall see (page 136) how these two rules can be combined into one rule by using negative exponents.

III. Law for Raising to Powers. $(x^a)^b = x^{ab}$

This is a special case of multiplication because

$$(x^a)^b = x^a \cdot x^a \cdot x^a \cdots \quad (b \text{ factors}) \quad = x^{a \text{ times } b}$$

IV. Law for Finding Roots. $\sqrt[b]{x^a} = x^{\frac{a}{b}}$

Since, by Law III, $(x^{\frac{a}{b}})^b = x^a$, therefore $x^{\frac{a}{b}} = \sqrt[b]{x^a}$.

Here a must be exactly divisible by b; but this restriction can be removed by using fractional exponents (page 137).

The definitions of the new kind of exponents are so made that the exponents obey the fundamental laws but permit more freedom by removing the restrictions.

153. Meaning of Zero as an Exponent. In order that Law I may be true for an expression like a^0 it must be true that

$$a^0 \cdot a^m = a^{0+m} = a^m$$

Dividing this equation by a^m (which we can do unless $a = 0$) we get $\quad a^0 = 1$

Hence any number (except zero) with an exponent zero equals 1. Thus, $4^0 = 1$; $(-6)^0 = 1$; and $(2x - 7)^0 = 1$ unless $x = 3.5$.

154. ORAL EXERCISES

State the numerical values of:

1. $2^0 \qquad 3 \cdot 2^0 \qquad (3 \cdot 2)^0 \qquad 6^0 \cdot 0 \qquad 0^3 \cdot 6^0$

2. $(-3)^0 \qquad -3^0 \qquad (\frac{2}{3})^0 \qquad \dfrac{2}{3^0} \qquad \dfrac{2^0}{3}$

3. State the product of: $a^x \cdot a^0$; $\quad e^x \cdot e^0$; $\quad 10^2 \cdot 10^0$

State simpler forms of the following expressions:

4. $2x^0 \qquad (2x)^0 \qquad (-2x)^0 \qquad -2x^0 \qquad -(2x)^0$

5. $\dfrac{2}{x^0} \qquad \dfrac{2^0}{x} \qquad 2 + x^0 \qquad (2 + x)^0$

6. $\dfrac{1}{a^0 + a} \qquad \dfrac{1}{b^0} + \dfrac{1}{b} \qquad \dfrac{1}{(b^0 + b)} \qquad \dfrac{b^0}{b^0 + b}$

155. Meaning of a Negative Exponent. In order that the law of division may be true for *all* integers we must say:

$$a^3 \div a^5 = a^{3-5} = a^{-2}; \quad \text{but } a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}$$

This suggests the definition: a^{-2} is the same as $\frac{1}{a^2}$.

A similar argument for literal exponents is:

$$a^m \cdot a^{-m} = a^{m-m} = a^0 = 1 \qquad (\text{unless } a = 0)$$

Dividing the first and last members by a^m, we get

$$a^{-m} = \frac{1}{a^m}$$

Hence $4^{-1} = \frac{1}{4}$; $10^{-2} = \frac{1}{100}$; $(x+2)^{-3} = \frac{1}{(x+2)^3}$ unless $x = -2$.

Now $x^a \div x^b = x^{a-b}$ whether $a > b$ or $b > a$.

156. EXERCISES

1. Prove that $\frac{1}{a^{-m}} = a^m$.

State the numerical values of the following:

2. 3^{-2}

3. $(\frac{1}{3})^{-2}$

4. $10 \cdot 3^{-1}$

5. $(10 \cdot 3)^{-1}$

6. $10^{-1} \cdot 3$

7. $12 \cdot 3^{-1}$

8. $\frac{12}{3^{-1}}$

9. $\frac{12^{-1}}{3}$

10. $\frac{9^{-2}}{3^{-4}}$

11. 2.36×10^{-2}

12. 2.36×10^2

13. $.269 \times 10^{-3}$

14. $.269 \times 10^3$

15. $.003 \times 10^{-4}$

16. $.003 \times 10^4$

17. Show that $\frac{2^{-1}}{2^{-2} - 3^{-1}}$ equals -6

18. Show that $\frac{3}{a^{-2} + b^{-2}}$ equals $\frac{3a^2b^2}{b^2 + a^2}$

Write with positive exponents and simplify:

19. $\frac{a^{-2}}{a^{-2} - b^{-2}}$

20. $\frac{a^{-3} + b^{-3}}{a^{-1} + b^{-1}}$

21. $\frac{x^{-3}y^{-3}}{x^{-2} - y^{-2}}$

157. Meaning of a Fractional Exponent. If the law $x^a x^b = x^{a+b}$ is to be true for fractional exponents then

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1, \text{ or } x$$

Since $x^{\frac{1}{2}}$ multiplied by itself gives x, then $x^{\frac{1}{2}}$ must be just another way of writing the square root of x. Hence

$$4^{\frac{1}{2}} = \sqrt{4} = 2 \quad \text{and} \quad (9\,y^8)^{\frac{1}{2}} = \sqrt{9\,y^8} = 3\,y^4$$

Similarly, $x^{\frac{1}{3}}$ is another way of writing the cube root of x because

$$x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1, \text{ or } x$$

Thus, $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ and $(125\,y^6)^{\frac{1}{3}} = \sqrt[3]{125\,y^6} = 5\,y^2$

In the same way $x^{\frac{1}{4}} = \sqrt[4]{x}$ and $x^{\frac{1}{n}} = \sqrt[n]{x}$

Also $x^{-\frac{1}{2}} = \dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{\sqrt{x}}$ and $x^{-\frac{1}{3}} = \dfrac{1}{x^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{x}}$

158. EXERCISES

Read and state the values of:

1. $25^{\frac{1}{2}}$	**6.** $36^{\frac{1}{2}}$	**11.** $32^{\frac{1}{5}}$	**16.** $(\frac{1}{4})^{\frac{1}{2}}$
2. $25^{-\frac{1}{2}}$	**7.** $36^{-\frac{1}{2}}$	**12.** $64^{\frac{1}{6}}$	**17.** $(\frac{1}{8})^{\frac{1}{3}}$
3. $16^{\frac{1}{2}}$	**8.** $4^{\frac{1}{2}}$	**13.** $9^{\frac{1}{2}}$	**18.** $(\frac{1}{25})^{-\frac{1}{2}}$
4. $16^{-\frac{1}{2}}$	**9.** $8^{\frac{1}{3}}$	**14.** $9^{-\frac{1}{2}}$	**19.** $(.04)^{-\frac{1}{2}}$
5. $16^{\frac{1}{4}}$	**10.** $8^{-\frac{1}{3}}$	**15.** $81^{\frac{1}{4}}$	**20.** $.125^{-\frac{1}{3}}$

State the indicated roots:

21. $(27x^6)^{\frac{1}{3}}$ **22.** $(8\,a^6)^{\frac{1}{3}}$ **23.** $(25\,a^8)^{\frac{1}{2}}$ **24.** $(16\,b^6)^{-\frac{1}{2}}$

The expression $5\,x^{\frac{1}{2}}$ means 5 times the square root of x. The exponent $\frac{1}{2}$ applies, like any other exponent, only to the number x and not to the coefficient 5.

If $x = 36$, $y = \frac{1}{25}$, and $z = .008$, find the values of:

25. $x^{\frac{1}{2}}$	**28.** $y^{\frac{1}{2}}$	**31.** $z^{\frac{1}{3}}$	**34.** $x^{\frac{1}{2}}y^{\frac{1}{2}}$
26. $2\,x^{\frac{1}{2}}$	**29.** $3\,y^{\frac{1}{2}}$	**32.** $5\,z^{\frac{1}{3}}$	**35.** $y^{\frac{1}{2}}z^{\frac{1}{3}}$
27. $3\,x^{-\frac{1}{2}}$	**30.** $4\,y^{-\frac{1}{2}}$	**33.** $3\,z^{-\frac{1}{3}}$	**36.** $yz^{\frac{1}{3}}$

159. To find the meaning of $x^{\frac{3}{2}}$, we notice that

$$x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^3$$

so that $x^{\frac{3}{2}}$ means the cube of the square root of x.

It is true also that $x^{\frac{3}{2}}x^{\frac{3}{2}} = x^3$ so that $x^{\frac{3}{2}}$ multiplied by itself equals x^3; hence we can also say that

$$x^{\frac{3}{2}} \text{ is the square root of } x^3, \text{ or } x^{\frac{3}{2}} = \sqrt{x^3}$$

In general, $x^{\frac{p}{r}} = \sqrt[r]{x^p}$ and also $x^{\frac{p}{r}} = (\sqrt[r]{x})^p$.

For a fractional exponent, the pupil should remember that:

The denominator indicates the root.

The numerator indicates the power.

As we saw with the expression $x^{\frac{3}{2}}$, it is immaterial whether we find the root first and then raise the result to the power, or find the power first and then the root. When evaluating an expression, it is usually better to find the root first, as then we deal with smaller numbers.

160. EXERCISES

Write and read with fractional exponents:

1. $\sqrt{a^3}$ $\quad \sqrt[3]{a^2}$ $\quad \sqrt[4]{a^3}$ $\quad \sqrt[3]{a^4}$
2. $\sqrt[3]{x^4}$ $\quad \sqrt{b^5}$ $\quad \sqrt[4]{y^2}$ $\quad \sqrt[6]{y^3}$

Using a radical sign, write and read:

3. $a^{\frac{2}{3}}$ $\quad b^{\frac{3}{4}}$ $\quad x^{\frac{4}{3}}$ $\quad y^{\frac{3}{4}}$
4. $b^{\frac{3}{2}}$ $\quad r^{\frac{5}{2}}$ $\quad a^{\frac{1}{2}}$ $\quad x^{\frac{2}{5}}$

Find the numerical value of:

5. $4^{\frac{3}{2}}$
6. $8^{\frac{2}{3}}$
7. $25^{\frac{3}{2}}$
8. $8^{\frac{5}{3}}$
9. $9^{\frac{3}{2}}$
10. $4^{-\frac{3}{2}}$
11. $64^{-\frac{2}{3}}$
12. $25^{-\frac{3}{2}}$
13. $9^{-\frac{3}{2}}$
14. $81^{-\frac{3}{2}}$
15. $\left(\frac{1}{4}\right)^{\frac{3}{2}}$
16. $\left(\frac{1}{8}\right)^{\frac{2}{3}}$
17. $\left(\frac{4}{25}\right)^{\frac{3}{2}}$
18. $\left(\frac{9}{4}\right)^{-\frac{3}{2}}$
19. $\left(\frac{64}{27}\right)^{\frac{2}{3}}$
20. $\left(\frac{100}{9}\right)^{-\frac{3}{2}}$

Using radicals, write the following in reduced form :

21. $a^{\frac{4}{3}}$ (Work thus : $a^{\frac{4}{3}} = a^{\frac{3}{3}}a^{\frac{1}{3}} = a\sqrt[3]{a}$.)

22. $a^{\frac{5}{3}}$ $\quad b^{\frac{3}{2}}$ $\quad x^{\frac{5}{2}}$ $\quad y^{\frac{5}{3}}$

23. $r^{\frac{7}{3}}$ $\quad s^{\frac{7}{2}}$ $\quad a^{\frac{9}{2}}$ $\quad b^{\frac{5}{4}}$

24. $\left(\frac{a}{b}\right)^{\frac{3}{2}}$ $\quad \left(\frac{x}{y}\right)^{\frac{4}{3}}$ $\quad \left(\frac{r}{s}\right)^{\frac{5}{3}}$ $\quad \left(\frac{r}{t}\right)^{\frac{5}{2}}$

Since $(x^a)^b = x^{ab}$, we see that $(x^{\frac{3}{4}})^2 = x^{\frac{3}{2}}$ and $(x^6)^{\frac{2}{3}} = x^4$

Similarly, find the indicated powers of the following:

25. $(x^2)^3$ $\quad (x^{\frac{2}{3}})^3$ $\quad (x^{\frac{3}{4}})^8$ $\quad (x^{\frac{2}{3}})^6$

26. $(y^6)^{\frac{2}{3}}$ $\quad (y^{\frac{4}{3}})^6$ $\quad (y^6)^{\frac{4}{3}}$ $\quad (y^{\frac{1}{3}})^{\frac{3}{2}}$

27. $(r^{\frac{2}{3}})^{\frac{3}{2}}$ $\quad (r^{\frac{2}{3}})^6$ $\quad (r^{\frac{3}{4}})^{\frac{4}{3}}$ $\quad (r^{\frac{1}{3}})^{\frac{5}{2}}$

Since $x^ax^b = x^{a+b}$, we see that $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} = x^{\frac{5}{6}}$

Similarly, multiply the following:

28. $x^{\frac{1}{2}} \cdot x^{\frac{3}{2}}$ $\quad x^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}$ $\quad x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$

29. $\sqrt{y}\sqrt[3]{y}$ $\quad \sqrt[3]{y}\sqrt[3]{y}$ $\quad \sqrt{y}\sqrt[4]{y}$

30. $\sqrt[3]{r}\sqrt[4]{r}$ $\quad \sqrt[3]{r^2}\sqrt{r}$ $\quad \sqrt[3]{r^2}\sqrt{r^3}$

Since $x^a \div x^b = x^{a-b}$, we see that $x^{\frac{5}{3}} \div x^{\frac{1}{3}} = x^{\frac{4}{3}}$, or, written as a fraction :

$$\frac{x^{\frac{5}{3}}}{x^{\frac{1}{3}}} = x^{\frac{4}{3}}$$

Similarly, find the quotients of the following, using either positive or negative exponents :

31. $\dfrac{x^{\frac{5}{2}}}{x^{\frac{1}{2}}}$ $\quad$ **34.** $\dfrac{b^{\frac{4}{3}}}{b}$ $\quad$ **37.** $\dfrac{r^3}{r^{\frac{1}{2}}}$

32. $\dfrac{a^{\frac{1}{3}}}{a^{\frac{4}{3}}}$ $\quad$ **35.** $\dfrac{b^{\frac{1}{3}}}{b^2}$ $\quad$ **38.** $\dfrac{r^2}{r^{\frac{3}{4}}}$

33. $\dfrac{b^{\frac{3}{2}}}{b^{\frac{2}{3}}}$ $\quad$ **36.** $\dfrac{c^{\frac{2}{3}}}{c^3}$ $\quad$ **39.** $\dfrac{r^{\frac{1}{3}}}{r}$

161. EXERCISES

1. Solve the equation $x^{\frac{2}{3}} = 16$.

SUGGESTION. First cube both members of the equation, getting $x^2 = 16^3$. Next find the square root of each member: $x = \pm\sqrt{16^3}$, or $\pm 16^{\frac{3}{2}}$. Hence $x = \pm 64$.

Check both values of x in $x^{\frac{2}{3}} = 16$.

Similarly, solve the following equations:

2. $x^{\frac{3}{2}} = 8$
3. $x^{\frac{2}{3}} = 4$
4. $x^{\frac{3}{2}} = 27$
5. $x^{-\frac{3}{2}} = 27$
6. $x^{\frac{1}{3}} = 5$
7. $2x^{\frac{1}{3}} = 5$
8. $x^{-\frac{1}{2}} = 3$
9. $5x^{-\frac{1}{2}} = 3$
10. $x^{\frac{3}{4}} = 27$
11. $x^{-\frac{1}{3}} = 2$
12. $x^{\frac{2}{3}} = 4^{-1}$
13. $x^{\frac{3}{2}} = 8^{-2}$

14. The number 514 can be written as 5.14×10^2. The number 51400 equals 5.14 times what power of 10?

Also $.514 = 5.14 \times 10^{-1}$ and $.0514 = 5.14 \times 10^{-2}$. The number .00514 equals 5.14 times what power of 10?

15. The above exercise illustrates the following idea:

In scientific work, when a number is either large or small it is convenient to write the number as

(a number between 1 and 10) times (a power of 10).

Notice that:

$51.4 = 5.14 \times 10^1$	$.514 = 5.14 \times 10^{-1}$
$514. = 5.14 \times 10^2$	$.0514 = 5.14 \times 10^{-2}$
$5140. = 5.14 \times 10^3$	$.00514 = 5.14 \times 10^{-3}$

After studying the above numbers, invent some rules for telling easily what the exponent of 10 should be.

Write the following numbers as shown in ex. 15:

16. 14,230,000 (Young's modulus for brass.)
17. .00001653 (Coefficient of expansion for zinc.)
18. 92,897,000 (Distance in miles from earth to sun.)
19. 2,765 (The diameter of Mercury in miles.)
20. 43,560 (The number of square feet in an acre.)

162. EXERCISES FOR DRILL

Perform the indicated multiplications:

1. $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x - x^{\frac{3}{2}})$

2. $a^{\frac{1}{2}}x^{\frac{1}{2}}(a^{\frac{1}{2}} - x^{\frac{1}{2}})$

3. $x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})$

4. $x^{\frac{2}{3}}(x^{\frac{1}{3}} - x^{\frac{2}{3}} + x)$

5. $x^{\frac{4}{3}}(x^{\frac{1}{3}} - x^{-\frac{1}{3}})$

6. $(e^x + e^{-x})^2$

7. $(e^x + e^{-x})^3$

8. $(e^x - e^{-x})^2$

9. $(e^x - e^{-x})^3$

10. $(e^x + e^{-x})(e^x - e^{-x})$

11. $(a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b)(a - a^{\frac{1}{2}}b^{\frac{1}{2}} + b)$

12. $(r^{\frac{1}{3}} + s^{\frac{1}{3}})(r^{\frac{2}{3}} - r^{\frac{1}{3}}s^{\frac{1}{3}} + s^{\frac{2}{3}})$

13. $(r^{\frac{2}{3}} - s^{\frac{2}{3}})(r^{\frac{4}{3}} + r^{\frac{2}{3}}s^{\frac{2}{3}} + s^{\frac{4}{3}})$

14. $(a^{\frac{1}{2}} + b^{\frac{1}{2}}c^{-1})(a^{\frac{1}{2}} - b^{\frac{1}{2}}c^{-1})$

15. $(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})$

State the indicated quotients:

16. $x^{\frac{2}{3}} \div x^{\frac{1}{4}}$

17. $x^{\frac{3}{2}} \div x^{\frac{1}{3}}$

18. $y^{\frac{5}{6}} \div y^{\frac{2}{3}}$

19. $\sqrt[4]{a} \div \sqrt[3]{a}$

20. $\sqrt[3]{a^4} \div \sqrt[4]{a^3}$

21. $(x + x^{\frac{3}{2}} + x^2) \div x^{\frac{3}{2}}$

22. $(y^2 - 2 - y^{-2}) \div y^{-3}$

23. $(y^{\frac{4}{3}} + y^{\frac{2}{3}} + 2) \div y^{-\frac{1}{3}}$

24. $(a^2 + 4 + a^{-2}) \div a^2$

25. $(r + 2 + r^{-1}) \div r^{-1}$

As on page 22, find by division the quotients:

26. $(x^{\frac{4}{3}} - x - 2\,x^{\frac{2}{3}} + 3\,x^{\frac{1}{3}} - 3) \div (x^{\frac{2}{3}} - 3)$

27. $(a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$

28. $(r^4 + 4\,r^2 + 6 + 7\,r^{-2} + 2\,r^{-4}) \div (r^2 + 1 + 2\,r^{-2})$

29. $(e^{3x} + 5\,e^{2x} + 6\,e^x + 2) \div (e^x + 1)$

30. $(e^{3x} + e^x + e^{-x} + e^{-3x}) \div (e^x + e^{-x})$

31. Eliminate the negative exponents in $\dfrac{a^{-2}b^{-2}}{a^{-1} + b^{-3}}$ by multiplying the numerator and the denominator by a^2b^3.

32. Using the method of ex. 31, work ex. 18 to 21, page 136.

163. Historical Note. In reading the history of various discoveries and inventions in mathematics, the pupil should bear in mind that before the invention of the printing press or any of the wonderful means of rapid communication that we now have, it was very difficult for scientists or mathematicians to exchange information or ideas. Hence it is not always possible to say who deserves the most credit for some of the important work of the Middle Ages. We say that Nicole Oresme, a bishop in Normandy, in the fourteenth century first conceived a notation for fractional exponents, and Simon Stevin of Bruges (1548–1620) invented our present way of writing powers and introduced fractional exponents.

It was left for John Wallis (1616–1703), a professor of geometry at Oxford, to advance the use of fractional and negative exponents. As might be expected, the usefulness of the notation led at once to the solution of many new problems in various fields of mathematics, and helped Isaac Newton to his discovery of a means of writing any power of a binomial like $(a + b)^n$. Newton's method, which is now called the *Binomial Theorem*, will be studied in Chapter XVI. Among other things, we shall see how this theorem is used to compute very quickly and easily square roots, cube roots, or fractional powers of numbers.

Just as the pupil finds it easier to multiply $\sqrt[3]{x^2} \cdot \sqrt[4]{x^3}$ after rewriting the quantities with fractional exponents, so many of the processes of mathematics were made easier just by the invention of a clever way of expressing an idea.

NOTE. The chapter on logarithms (pages 221 to 239) may well be studied next, or at any time after the chapter on fractional exponents.

CHAPTER X

IMAGINARIES

164. Kinds of Numbers. Before introducing a new kind of number, called *imaginary number*, let us review the kinds of numbers that we have used so far.

When we first learned to count, only whole numbers, called integers, were used. Fractions were next introduced through the process of dividing one integer by another.

Negative, or minus, numbers were introduced in the work of subtracting a number from a smaller one.

Any number that can be written as the quotient of two integers is called a *rational number*.

Thus, 5, -4, $\frac{3}{2}$, and 7.6 are rational numbers.

In finding some square roots we met a new kind of number. The square root of 3, for example, is not rational because no matter how far we carry out the process of finding the root we never reach the end of the decimals. This is an example of an *irrational number;* it is a number that cannot be written as the quotient of two integers.

Thus, $\sqrt{6}$ and $\sqrt[3]{4}$ are irrational numbers. The fact that there is no end to the decimals is not a proof that a number is irrational. Repeating decimals like .3333 ⋯ can be expressed as fractions and are therefore rational. (See page 209, example 2.)

Both rational and irrational numbers are called *real numbers* to distinguish them from another kind of number that we first meet when we try to find the square root of a negative number. For example, $\sqrt{-4}$ is neither $+2$ nor -2 as we can easily see by squaring these numbers.

The expression $\sqrt{-4}$ represents a new kind of number.

165. Imaginary Numbers. An indicated square root of a negative number is called an *imaginary number*.

Because of the name, *imaginary*, the pupil should not think that such numbers are less practical than the numbers that we call *real*. Even fractions are impractical in some problems, such as for counting the number of people in a room; negative numbers are impractical when measuring the length of a rectangle. Different kinds of numbers have different uses. Many valuable discoveries in electrical engineering would have been impossible without the use of the so-called imaginary numbers.

When working with imaginary numbers, it is convenient to express them as some real number times $\sqrt{-1}$. Thus:

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} \qquad \sqrt{-5} = \sqrt{5}\sqrt{-1}$$
$$\sqrt{-a^2} = \sqrt{a^2}\sqrt{-1} = a\sqrt{-1} \qquad \sqrt{-c} = \sqrt{c}\sqrt{-1}$$

The letter i is used to represent the number $\sqrt{-1}$. The above examples are then written as:

$$\sqrt{-4} = 2i, \ \sqrt{-a^2} = ai, \ \sqrt{-5} = i\sqrt{5}, \ \sqrt{-c} = i\sqrt{c}$$

166. Addition. Imaginary numbers can be combined thus:

$$5\sqrt{-1} + 3\sqrt{-1} = 8\sqrt{-1}, \text{ or } 5i + 3i = 8i$$
$$\sqrt{-36} - \sqrt{-4} = 6\sqrt{-1} - 2\sqrt{-1}, \text{ or } 6i - 2i = 4i$$
$$\sqrt{-7} + \sqrt{-3} = i\sqrt{7} + i\sqrt{3}, \text{ or } i(\sqrt{7} + \sqrt{3})$$

A real number and an imaginary number cannot be *actually* added. We can only indicate the addition by writing such expressions as $5 + \sqrt{-3}$, $4 - 7i$, $2 + i\sqrt{3}$.

The indicated sum of a real and an imaginary number is called a *complex number*.

When adding several expressions composed of real and imaginary numbers, add the real numbers and the imaginary numbers separately. Thus:

$$6 - \sqrt{-4} + 7 + \sqrt{-9} = 6 - 2i + 7 + 3i = 13 + i$$
$$7 + 2i + 9 - 6i - 5 = 11 - 4i$$

167. EXERCISES

Express the following quantities as multiples of i; that is, read them as the product of some number and i:

1. $\sqrt{-6}$ It is better to say "i times the square root of 6" than "the square root of 6 times i." Why?

2. $\sqrt{-5}$	7. $3\sqrt{-25}$	12. $2\sqrt{-y^2}$
3. $\sqrt{-4}$	8. $3\sqrt{-\frac{1}{4}}$	13. $\sqrt{-3y^2}$
4. $\sqrt{-16}$	9. $5\sqrt{-\frac{1}{9}}$	14. $-3\sqrt{-y^2}$
5. $\sqrt{-9}$	10. $\sqrt{-a^2}$	15. $-\sqrt{-x^2}$
6. $2\sqrt{-9}$	11. $\sqrt{-x^2}$	16. $\sqrt{-x^2-y^2}$

Write the following quantities, using the symbol i, and then add any terms that can be added:

17. $\sqrt{-9}+\sqrt{-4}$	22. $\sqrt{-a^2}+\sqrt{-b^2}$
18. $\sqrt{-9}-\sqrt{4}$	23. $\sqrt{-a^2}-\sqrt{b^2}$
19. $\sqrt{-25}+\sqrt{-4}$	24. $\sqrt{a^2}-\sqrt{-a^2}$
20. $\sqrt{25}+\sqrt{-4}$	25. $x+\sqrt{-x^2}$
21. $\sqrt{-4}+\sqrt{4}$	26. $-x+\sqrt{-x^2}$

27. $(-9)^{\frac{1}{2}}+(-16)^{\frac{1}{2}}-(-25)^{\frac{1}{2}}+9^{\frac{1}{2}}$

28. $(-x^2)^{\frac{1}{2}}+(-4x^2)^{\frac{1}{2}}+(-9x^2)^{\frac{1}{2}}$

168. Powers of *i*. Since any imaginary number can be written as some real number times the number i, it will be useful to record the powers of i.

$$\begin{array}{llll}
\text{By definition } i^2 = & & & -1 \\
\text{Then} \quad i^3 = i^2 \cdot i & = -1 \cdot i & & = -i \\
\phantom{\text{Then}} \quad i^4 = i^2 \cdot i^2 & = -1 \cdot -1 & & = +1 \\
\phantom{\text{Then}} \quad i^5 = i^4 \cdot i & = +1 \cdot i & & = +i
\end{array}$$

Continuing this process we find that the powers always reduce to one of the four values: i, -1, $-i$, $+1$.

Find and compare the values of i^5, i^6, i^7, and i^8.

169. Multiplication. The product $\sqrt{-2}\sqrt{-3}$ cannot be found by using Rule 1 on page 122 because that rule applies only to real numbers. The correct product is found thus:

$$\sqrt{-2} = i\sqrt{2} \quad \text{and} \quad \sqrt{-3} = i\sqrt{3}$$

Hence $\sqrt{-2}\sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = i^2\sqrt{6} = -1\sqrt{6}$

$$= -\sqrt{6}$$

In general, $\sqrt{-a}\sqrt{-b} = i\sqrt{a} \cdot i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}$.

Either the pupil must remember this rule or, better, in all such problems the pupil should *write each number in terms of* i, and remember that $i^2 = -1$.

Notice the difference in the following products:

1. $\sqrt{-3}\sqrt{-7} = i\sqrt{3} \cdot i\sqrt{7} = i^2\sqrt{21} = -\sqrt{21}$
2. $\sqrt{3} \quad \sqrt{-7} = \sqrt{3} \cdot i\sqrt{7} = i\sqrt{21} = \sqrt{-21}$

170. To multiply two complex numbers, write each number in terms of i and then proceed as on page 18:

EXAMPLE. Multiply $2 + \sqrt{-3}$ by $4 - \sqrt{-5}$.

$$\begin{array}{rl}
2 + \sqrt{-3} = & 2 + \quad i\sqrt{3} \\
4 - \sqrt{-5} = & \underline{4 - \quad i\sqrt{5}} \\
& 8 + 4\,i\sqrt{3} \\
& \underline{\quad - 2\,i\sqrt{5} \qquad\qquad - i^2\sqrt{15}} \\
& 8 + i(4\sqrt{3} - 2\sqrt{5}) + \sqrt{15}
\end{array}$$

171. Division. The fraction $\dfrac{\sqrt{6}}{\sqrt{-2}}$ indicates the division of $\sqrt{6}$ by $\sqrt{-2}$. To make the divisor a real number, multiply the numerator and the denominator by $\sqrt{-2}$. Thus:

$$\frac{\sqrt{6}}{\sqrt{-2}} = \frac{\sqrt{6} \cdot \sqrt{-2}}{\sqrt{-2} \cdot \sqrt{-2}} = \frac{i\sqrt{12}}{-2} = \frac{2\,i\sqrt{3}}{-2} = -\,i\sqrt{3}$$

Likewise

$$\frac{5}{2 + \sqrt{-3}} = \frac{5 \qquad (2 - i\sqrt{3})}{(2 + i\sqrt{3})(2 - i\sqrt{3})} = \frac{10 - 5\,i\sqrt{3}}{7}.$$

172. EXERCISES

Perform the indicated multiplications and simplify the results as far as possible:

1. $\sqrt{-3}\sqrt{-3}$
2. $\sqrt{-9}\sqrt{-4}$
3. $\sqrt{-9}\sqrt{4}$
4. $\sqrt{9}\sqrt{-4}$
5. $\sqrt{-a^2}\sqrt{-b^2}$
6. $\sqrt{a^2}\sqrt{-b^2}$
7. $\sqrt{-a^2}\sqrt{b^2}$
8. $\sqrt{-x}\sqrt{-xy^2}$
9. $\sqrt{-\frac{1}{2}}\sqrt{-\frac{1}{3}}$
10. $(5+i)(2+i)$
11. $(5+i)(5-i)$
12. $(6-5i)(2+3i)$
13. $(x+iy)(x-iy)$
14. $(3+\sqrt{-1})(3-\sqrt{-1})$
15. $(2+\sqrt{-3})(5+\sqrt{-3})$
16. $(2+\sqrt{-3})(5-\sqrt{3})$
17. $(5+\sqrt{-3})^2$
18. $(7-\sqrt{-5})^2$
19. $(a+bi)^2+(a-bi)^2+(a+bi)(a-bi)$

Make the denominators of the following fractions into real and rational numbers:

20. $\dfrac{\sqrt{18}}{\sqrt{-3}}$
21. $\dfrac{\sqrt{-18}}{\sqrt{-3}}$
22. $\dfrac{\sqrt{-18}}{\sqrt{3}}$
23. $\dfrac{\sqrt{-20}}{\sqrt{-5}}$
24. $\dfrac{3}{2+\sqrt{-1}}$
25. $\dfrac{4+\sqrt{-5}}{4-\sqrt{-5}}$

26. Show by substitution that $x = 2+\sqrt{-2}$ is a solution of the equation $x^2 - 4x + 6 = 0$.

27. Show that $x = 3 + i$ and $x = 3 - i$ are both solutions of the equation $x^2 - 6x + 10 = 0$.

28. Every number has *two square roots*, and *three cube roots*. To find all the cube roots of 1, for example, write the equation $x^3 = 1$, or $x^3 - 1 = 0$. Solve this equation by factoring the left member: $(x-1)(x^2+x+1) = 0$.

Two of the solutions are imaginary numbers. Denoting them by A and B, prove the following interesting facts:

(a) $A^2 = B$ (b) $B^2 = A$ (c) $AB = 1$

173. REVIEW OF CHAPTERS VII TO X

Solve the following equations and sets of equations:

1. $\dfrac{2x-1}{x+2}+\dfrac{3x-10}{3-x}=\dfrac{11-x-x^2}{x^2-x-6}$

2. $\dfrac{x-3}{x+3}-\dfrac{x^2-24}{x^2+6x+9}=\dfrac{x^2+9x+5}{x^3+6x^2+9x}$

3. $\dfrac{2y+1.1}{.3}-\dfrac{y-.05}{.2}=\dfrac{4y+.3}{.5}-16$

4. $\dfrac{3x-.5}{.2}+\dfrac{2x+.25}{.5}-\dfrac{2(x-1.5)}{2.5}=44.7$

5. $\dfrac{x-a}{a-1}+\dfrac{x-a}{a+1}=2a$

6. $\dfrac{x-2b^2}{a-b}-\dfrac{x-10b^2}{a+3b}=4b$

7. $\begin{cases} 2x+2y=3(a+b) \\ bx-ay=b^2-a^2 \end{cases}$

8. $\begin{cases} x+2y-3z=5 \\ 2x+y+6z=2 \\ x+4y+3z=8 \end{cases}$

9. $\begin{cases} x+2y-z=3a \\ 2x-y+z=a+b \\ 3x-3y+2z=b \end{cases}$

10. $\begin{cases} x+y+z+u=2 \\ x-y+z+u=-2 \\ x+y-z+u=6 \\ x+y+z-u=-4 \end{cases}$

11. $\begin{cases} x+y+z+u=5 \\ x-y+2z=6 \\ y-z+3u=6 \\ x+2y-2u=-7 \end{cases}$

12. $\begin{cases} ax+b^2y=a^2+2ab \\ 3ax-2b^2y=3a^2+ab \end{cases}$

13. $\begin{cases} x-y+3z=0 \\ 2x+2y+z=-\frac{5}{3} \\ 3x+y-z=0 \end{cases}$

14. $\begin{cases} x+y=z \\ bx+ay=2b(z-b) \\ ax-by=z(a-b) \end{cases}$

15. $\begin{cases} x+y+z=0 \\ y+z+w=-2 \\ x+y+w=2 \\ x-z+w=3 \end{cases}$

16. $\begin{cases} x+2y+z=3 \\ x-2y+3z+w=4 \\ 2x-4y+w=5 \\ 6y-z+w=6 \end{cases}$

174. REVIEW — EXPONENTS AND RADICALS

1. What is the quotient of $x^{a^2-b^2} \div x^{a-b}$?

2. What is the product of $a^{x-y} \cdot a^{x+y}$?

3. Is $x^{-\frac{1}{4}}$ or $x^{\frac{1}{4}}$ the square root of $x^{-\frac{1}{2}}$?

4. What is the square root of x^{2a}? of a^{x^2}?

5. What power of 2 does $\sqrt[3]{2}\sqrt[6]{2}$ equal?

6. Show that $3^7 \cdot 9^n \cdot 81^{n+1} = 3^{6n+11}$. (Change 9^n to $(3^2)^n$, or 3^{2n}; and change 81^{n+1} to $(3^4)^{n+1}$, or 3^{4n+4}. Then $3^7 \cdot 3^{2n} \cdot 3^{4n+4}$ = what power of 3?)

7. Express as a power of 3: $3^n \cdot 9^{n-1} \cdot 27^{2n}$.

8. Express as a power of 2: $2^6 \cdot 4^n \cdot 8^{n+1}$.

9. Express as a power of 4: $2^8 \cdot 16^n \cdot 32^{2n}$.

10. If $2^3 \cdot 8^{n+1} \cdot 4^n = 2^{16}$ find n. (Change the given equation to $2^3 \cdot 2^{3n+3} \cdot 2^{2n} = 2^{16}$. Then it must be true that $3 + 3n + 3 + 2n = 16$. Solve for n.)

As in ex. 10, solve the following equations for n:

11. $2^n \cdot 4^{n-1} \cdot 8^{n+1} = 2^7$

12. $3 \cdot 9^{n+2} \cdot 27^{4+n} = 9$

13. $4^n \cdot 8^{n-2} 16^n = 8$

14. $9^2 \cdot 3^{n+1} \cdot 27^n = 3^7$

15. $5 \cdot 25^n \cdot 625^{2n} = 5^{11}$

16. $5^2 \cdot 125^n = 5^{n+6}$

17. Find the value of $x^2 - 8x + 17$ when $x = 4 + i$.

18. Find the value of $x^2 - 2ax$ when $x = a + ai$.

19. Since $(a + 2i)(a - 2i) = a^2 + 4$, what are the factors of $a^2 + 4$?

20. Using imaginary numbers, as in ex. 19, factor:

(a) $a^2 + b^2$ (b) $x^2 + 4y^2$ (c) $a^2 + 9b^2$

CHAPTER XI

QUADRATIC EQUATIONS

175. Review of Trinomial Squares. A method of solving quadratic equations by factoring was studied on page 57. Another method, which is used when factoring is impossible, depends on using trinomials that are squares. This method is particularly important because it leads to a formula by means of which all quadratics can be solved rapidly.

A trinomial like $9x^2 + 30xy + 25y^2$ whose two factors, $(3x + 5y)(3x + 5y)$, are the same is called a trinomial square. Either factor, since both are the same, is the square root of the trinomial.

Such squares can be recognized at sight because two terms must be squares and the other term twice the product of the square roots of the terms that are squares.

Thus, $9x^2 + 30xy + 25y^2$ is a square because:

(1) The first term, $9x^2$, is the square of $3x$.

(2) The last term, $25y^2$, is the square of $5y$.

(3) The middle term, $30xy$, is twice the product of these square roots. $30xy$ equals $2 \cdot 3x \cdot 5y$.

Note that since the square root of $9x^2$ is either $+3x$ or $-3x$ and the square root of $25y^2$ is either $+5y$ or $-5y$, the middle term might be either $+30xy$ or $-30xy$, and the trinomial would still be a square.

$$9x^2 - 30xy + 25y^2 = (3x - 5y)^2$$

176. ORAL EXERCISES

Which of the following trinomials are squares? State their square roots. What changes are needed to make the others into trinomial squares?

1. $4a^2 + 12ab + 9b^2$
2. $4a^2 - 12ab + 9b^2$
3. $9a^2 + 25b^2 + 16c^2$
4. $4x^2 - 10xy + 25y^2$
5. $24x + 16 + 9x^2$
6. $100x^2 + 180xy + 81y^2$
7. $100 + 81y^2 - 180xy$
8. $25x^2 - 54xy + 81y^2$
9. $9x^2 + 90xy + 100y^2$
10. $9a^2 + 60ab + 24b^2$

Find the missing term in each of the following so that the resulting trinomial will be a square:

11. $x^2 + \qquad + 16$
12. $y^2 - \qquad + 9$
13. $4a^2 + \qquad + 49b^2$
14. $x^2 + \qquad + 64$
15. $x^2 + \qquad + \frac{1}{9}$
16. $x^2 + \qquad + \frac{1}{4}$

17. If x^2 and $+10x$ are the first and second terms of a square, the missing third term can be found as follows:

According to the rules, the middle term, $+10x$, is *twice* the product of certain square roots. Hence $+5x$ is the product of the square roots. The square root of the first term, x^2, is x; hence the other square root is 5. The third term is therefore 5^2, or 25. Thus:

$x^2 + 6x +$	The missing third term is $+9$.
$x^2 - 12x +$	The missing third term is $+36$.
$a^2 - 14ab +$	The missing third term is $+49b^2$.

What are the terms needed to make the following quantities squares?

18. $x^2 + 16x +$
19. $a^2 + 10ab +$
20. $y^2 - 6y +$
21. $r^2 + 8r +$
22. $s^2 + 14s +$
23. $t^2 - 12t +$
24. $t^2 + 12t +$
25. $x^2 + 3x +$
26. $c^2 + 5c +$
27. $y^2 + 7y +$
28. $a^2 - 9a +$
29. $b^2 + 11b +$
30. $x^2 + x +$
31. $x^2 - x +$
32. $x^2 + ax +$

177. Solution by Completing the Square.

EXAMPLE 1. Solve the equation $x^2 + 6x - 16 = 0$.

First transpose -16: $x^2 + 6x = 16$

Add 9 to each member, because this will make the left member a square: $x^2 + 6x + 9 = 25$

Find the square root of each member: $x + 3 = 5$

From this new equation, find x: $x = 2$

This solution is not complete because a quadratic equation has two solutions. The second solution can also be found from the above work if we notice that the square root of 25 is either $+5$ or -5. Hence

$$\left.\begin{aligned} x + 3 &= 5 \\ x &= 2 \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} x + 3 &= -5 \\ x &= -8 \end{aligned}\right.$$

EXAMPLE 2. Solve the equation $x^2 - 10x + 9 = 0$.

Transpose $+9$: $x^2 - 10x = -9$

Add $+25$ to each member: $x^2 - 10x + 25 = 16$

Find the square root of each member: $x - 5 = \begin{cases} +4 \text{ or} \\ -4 \end{cases}$

Solve the two new equations: $\begin{cases} x - 5 = 4 \\ x = 9 \end{cases} \begin{cases} x - 5 = -4 \\ x = 1 \end{cases}$

The two solutions $x = 9$ and $x = 1$ should next be checked by substituting them in the equation $x^2 - 10x + 9 = 0$.

An easy way of writing the two square roots of a number, 16 for example, is to write ± 4, meaning "$+4$ or -4."

The trinomial $x^2 - 10x + 25$ also has two square roots, one of which is the negative of the other. $(x - 5)$ is one square root and $-(x - 5)$ is the other; but if we write

$$-(x - 5) = 4 \quad \text{and} \quad -(x - 5) = -4$$

we find that $x = 1$ and $x = 9$ are also the solutions of these equations. There are always just two solutions.

RULES. *Write the equation in the form* $x^2 + bx = c$.

Add to both members the square of one half of b.

Find the square root of both members of the equation and use both square roots of the right member, thus obtaining two simpler equations. Solve the two new equations for x.

Check both solutions by substituting them in the original equation.

178. EXERCISES

The equations are arranged here in three groups, 1 to 12, 13 to 21, and 22 to 30, according to the care needed in the arithmetic work.

Solve the following equations by completing the square. Ex. 1 to 21 can be checked by solving them a second time by factoring. In ex. 22 to 30 factoring is not possible. Check these as in ex. 11, page 127.

1. $x^2 - 8x - 20 = 0$	**7.** $x^2 + 12x = 28$
2. $x^2 + 10x + 21 = 0$	**8.** $x^2 + 14x + 48 = 0$
3. $x^2 - 6x - 7 = 0$	**9.** $t^2 = 10t - 24$
4. $r^2 + 6r + 8 = 0$	**10.** $y^2 - 15 = 2y$
5. $y^2 - 12y + 20 = 0$	**11.** $v^2 = 16v - 15$
6. $t^2 - 2t - 3 = 0$	**12.** $36 + x^2 = 20x$

13. $y^2 - 9y + 20 = 0$ Think of $\frac{1}{2}$ of 9 as $\frac{9}{2}$, not as $4\frac{1}{2}$.

14. $x^2 - 7x - 8 = 0$	**18.** $t^2 = 4 - 3t$
15. $y^2 + 5y = 36$	**19.** $r^2 = 5r - 6$
16. $r^2 - 3r - 40 = 0$	**20.** $s^2 + 7s = -12$
17. $s^2 + 9s = 10$	**21.** $y^2 = 5y + 14$

22. $x^2 - 6x = 4$ ANS. $x = 3 \pm \sqrt{13}$

23. $x^2 - 6x = 2$	**27.** $x^2 + 6x = 6$
24. $x^2 + 8x = 5$	**28.** $x^2 - 6x = 6$
25. $y^2 = 10y - 18$	**29.** $x^2 + 6x = -6$
26. $12x - 3 = x^2$	**30.** $x^2 - 6x = -6$

31. One arm of a right triangle is 4 in. longer than the other arm. If the hypotenuse is 18 in. long, how long are the arms? (Use the table on page 271 to find square roots.)

32. A platform is 10 ft. wide and 20 ft. long. The owner decides to make the platform twice as large by increasing the width and the length the same amount. About how many feet does he add to each dimension?

179. The following solution needs careful study. In the rule on page 152, the coefficient of x^2 must be 1. Hence, to use the rule here, we first divide by the coefficient of x^2.

EXAMPLE. Solve the equation $5x^2 - 3x - 4 = 0$.

Divide by the coefficient of x^2: $\qquad x^2 - \frac{3x}{5} - \frac{4}{5} = 0$

Transpose $-\frac{4}{5}$: $\qquad x^2 - \frac{3x}{5} = \frac{4}{5}$

$\frac{1}{2}$ of $\frac{3}{5}$ is $\frac{3}{10}$; $\left(\frac{3}{10}\right)^2 = \frac{9}{100}$
Hence add $\frac{9}{100}$ to each member: $\qquad x^2 - \frac{3x}{5} + \frac{9}{100} = \frac{4}{5} + \frac{9}{100} = \frac{89}{100}$

Find the square roots: $\qquad x - \frac{3}{10} = \frac{\pm\sqrt{89}}{10}$

Transpose $-\frac{3}{10}$: $\qquad x = \frac{3}{10} \pm \frac{\sqrt{89}}{10}$

The solutions are usually written as $\frac{3 + \sqrt{89}}{10}$ and $\frac{3 - \sqrt{89}}{10}$. Unless other directions are given, the values in decimals need not be found.

180. EXERCISES

Solve and check the following equations:

1. $3x^2 - 5x - 2 = 0$	**11.** $4y^2 + 7y - 15 = 0$
2. $3x^2 + 5x - 2 = 0$	**12.** $2r^2 + 9r + 9 = 0$
3. $3x^2 - 5x - 4 = 0$	**13.** $4s^2 + 8s + 3 = 0$
4. $3x^2 - 4x - 4 = 0$	**14.** $6t^2 - 5t - 2 = 0$
5. $3x^2 - 4x + 1 = 0$	**15.** $6v^2 - v - 2 = 0$
6. $3x^2 - 4x - 2 = 0$	**16.** $3y^2 + y - 10 = 0$
7. $5x^2 + 8x - 4 = 0$	**17.** $3x + 4 = 2x^2$
8. $6x + 2 = 5x^2$	**18.** $5s = 2s^2 + 3$
9. $7x = 5x^2 + 1$	**19.** $t = 2t^2 - 1$
10. $12 = 4x + 5x^2$	**20.** $8y^2 - 16y + 3 = 0$

181. PROBLEMS — QUADRATIC EQUATIONS

1. What number added to its square equals 42? (We can guess that the number is 6, but there are two answers.)

2. The sum of 20 and the square of a certain number equals 9 times the number. Find two numbers for which this is true.

3. If 3 times the square of a number is increased by twice the number, the result is 8. Find the number.

4. If 5 times a certain number is added to the square of the number, the result is 66. Find the number.

5. The difference of two numbers is 5. The sum of their squares is 53. Find all the numbers for which this is true.

6. The sum of two numbers is 12. The square of the larger added to 2 times the square of the smaller is 99. Find all the numbers for which this is true.

7. What number added to one half its square equals 12?

Draw a figure for each of the exercises 8 to 14. In the figure write the length of each line either as a known number or as an expression in x.

8. Find the dimensions of a rectangle whose area is 54 sq. ft. if the sum of the base and the altitude is $16\frac{1}{2}$ ft.

9. Find the dimensions of a rectangle whose area is 243 sq. in. if the sum of the base and the altitude is 36 in.

10. The length of a rectangle is 3 ft. more than 2 times the width. If the area is 104 sq. ft., find its dimensions.

11. Find the two arms of a right triangle if its hypotenuse is 13 in. and one arm is 7 in. longer than the other.

Find the base and the altitude of a triangle if:

12. The altitude is 3 ft. less than the base and the area is 20 sq. ft.

13. The base is 5 in. longer than the altitude and the area is 42 sq. in.

14. The area is $49\frac{1}{2}$ sq. in. and the sum of the base and the altitude is 20 in.

182. Solving Literal Quadratics by Completing the Square. Quadratic equations that have literal numbers for the coefficients are solved like other equations, but require skill in using fractions and radicals.

EXAMPLE. Solve for x: $ax^2 + 3\,bx - 5\,b = 0$

Divide by the coefficient of x^2: $\quad x^2 + \frac{3\,b}{a}x - \frac{5\,b}{a} = 0$

Transpose the term $-\frac{5\,b}{a}$: $\quad x^2 + \frac{3\,b}{a}x = \frac{5\,b}{a}$

$\frac{1}{2}$ of $\frac{3\,b}{a}$ is $\frac{3\,b}{2\,a}$; add $\left(\frac{3\,b}{2\,a}\right)^2$, or $\frac{9\,b^2}{4\,a^2}$, to each member:

$$x^2 + \frac{3\,b}{a}x + \frac{9\,b^2}{4\,a^2} = \frac{9\,b^2}{4\,a^2} + \frac{5\,b}{a}$$

Write the left member as a square; simplify the right member:

$$\left(x + \frac{3\,b}{2\,a}\right)^2 = \frac{9\,b^2 + 20\,ab}{4\,a^2}$$

You cannot always find the square root of the right member, but you can indicate the square root:

$$x + \frac{3\,b}{2\,a} = \frac{\pm\sqrt{9\,b^2 + 20\,ab}}{2\,a}$$

The solutions are $\frac{-3\,b + \sqrt{9\,b^2 + 20\,ab}}{2\,a}$ and $\frac{-3\,b - \sqrt{9\,b^2 + 20\,ab}}{2\,a}$.

When further simplifications are possible they should be performed. A simple method of checking the answers will be studied on page 158.

183. EXERCISES

After studying the above example, solve for x:

1. $x^2 + 2\,ax - b = 0$
2. $x^2 + 6\,ax - c = 0$
3. $x^2 - 4\,ax + r = 0$
4. $2\,x^2 - 6\,ax = b$
5. $3\,x^2 - 12\,ax = c^2$
6. $3\,x^2 + 9\,bx + b^2 = 0$
7. $x^2 + 2\,ax = 2\,a + 1$
8. $x^2 + 6\,ax = 3\,a^2$
9. $x^2 + 2\,bx + ab = 0$
10. $ax^2 + 2\,bx + ab = 0$
11. $ax^2 + 2\,bx + c = 0$
12. $ax^2 + bx + c = 0$

184. A Formula for Solving Quadratic Equations. All quadratic equations can be written in what is called the *standard* or *general* form:

$$\boxed{ax^2 + bx + c = 0}$$

Here the letter a is the coefficient of x^2, and b is the coefficient of x, while c is the term that does not contain the unknown number.

Thus, for $3x^2 + 5x - 7 = 0$, we have $a = 3$, $b = 5$, $c = -7$.

By solving the standard equation by the method of completing the square we obtain a formula:

SOLUTION. Solve for x: $ax^2 + bx + c = 0$

Divide by a; transpose $\frac{c}{a}$: $x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\frac{1}{2}$ of $\frac{b}{a}$ is $\frac{b}{2a}$. Add $\left(\frac{b}{2a}\right)^2$, or $\frac{b^2}{4a^2}$, to each member:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

or
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Then
$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}}$$

To show how the formula is used let us apply it to the equation $3x^2 + 5x - 7 = 0$. Wherever a appears in the formula we write 3; wherever b appears we write 5; and wherever c appears we write -7. The result is

$$x = \frac{-5 \pm\sqrt{(5)^2 - 4(3)(-7)}}{2 \cdot 3} = \frac{-5 \pm\sqrt{25 + 84}}{6}$$

$$= \frac{-5 \pm\sqrt{109}}{6}$$

185. The Sum and the Product of the Roots. Hereafter we shall call the "solutions" of an equation its "roots." (See page 26.) This word has not been much used previously in order to avoid confusion with "square roots."

Calling the roots r_1 and r_2, we can write them as:

$$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

If we add the roots, we find that:

$$r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

If we multiply the two roots, we find that:

$$r_1 \cdot r_2 = \frac{(-b+\sqrt{b^2-4ac})}{2a} \cdot \frac{(-b-\sqrt{b^2-4ac})}{2a}$$
$$= \frac{b^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

These useful results should be remembered in the words:

The roots of $\boldsymbol{ax^2+bx+c=0}$ *are* $\boldsymbol{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$.

The sum of the roots is $-\frac{b}{a}$; *the product is* $\frac{c}{a}$.

186. *By finding the sum and the product of the roots we can easily check the roots of any quadratic equation.*

EXAMPLE. Solve and check: $3x^2 - 7x - 20 = 0$.

Here $a = 3$, $b = -7$, $c = -20$. Notice also that $b^2 = 49$, and that the term $-4ac = -4 \cdot 3 \cdot -20 = 240$. Then, by the formula:

$$x = \frac{7 \pm\sqrt{49+240}}{6} = \frac{7\pm\sqrt{289}}{6} = \frac{7 \pm 17}{6}$$

One root is $\frac{7+17}{6}$, or 4; the other root is $\frac{7-17}{6}$, or $-\frac{5}{3}$.

Check. The sum of the roots is $4-\frac{5}{3}$, or $\frac{7}{3}$, and $-\frac{b}{a} = \frac{7}{3}$.

The product of the roots is $4 \cdot -\frac{5}{3}$, or $-\frac{20}{3}$, and $\frac{c}{a} = -\frac{20}{3}$.

187. ORAL EXERCISES

Read each of the following equations in the standard form, rearranging the terms where necessary. State the values of a, b, and c in each equation. If a is negative, multiply all the terms by -1 to make a positive. Also state the sum of the roots and the product of the roots.

1. $3x^2 + 8 = 14x$	**6.** $2y^2 = 3y + 1$
2. $x^2 = 8x - 15$	**7.** $4 + 4t = 3t^2$
3. $13x - 15 = 2x^2$	**8.** $3s - 5s^2 + 2 = 0$
4. $9x^2 = 3x + 2$	**9.** $3z + 2 - z^2 = 0$
5. $15 - 7x = 2x^2$	**10.** $1 - 8r + 3r^2 = 0$

188. EXERCISES

1 to 10. Solve equations 1 to 10 above, using the formula. Check the roots by finding their sum and product.

Solve the following equations by any one of the three methods: factoring, completing the square, or by the formula. Check the roots in the *given* equation.

11. $9x(x + 2) - 3x(x - 2) = 2(x + 4)(x + 6)$

12. $(r + 2)^2 - (r - 2)^2 + (r - 2)(r + 2) = 44$

13. $2y^2 + (y - 4)^2 = 5y(y - 4) + 24$

14. $(4r - 7)^2 - 3(2r - 5)^2 = 47 - (6r - 5)^2$

15. $(x + \frac{1}{2})^2 - (x - \frac{1}{2})^2 = (x - \frac{1}{2})(x + \frac{1}{2}) + \frac{5}{4}$

16. $\dfrac{14}{x + 2} - \dfrac{1}{x - 4} = 1$

17. $\dfrac{3}{x - 4} - \dfrac{2}{x + 1} = 1$

18. $x + \dfrac{1}{x} = \dfrac{13}{6}$

19. $\dfrac{3}{y} + \dfrac{y}{3} = \dfrac{5}{2}$

20. $\dfrac{4}{x + 1} + \dfrac{7}{x + 2} = \dfrac{x^2 - 2x + 51}{x^2 + 3x + 2}$

21. $\dfrac{3}{x + 2} + \dfrac{2}{x - 4} = \dfrac{x + 2}{3x + 6}$

22. $\dfrac{x - 7}{x^2 - x} = \dfrac{2}{x - 1} - \dfrac{x - 2}{x}$

189. PROBLEMS

1. A boy planted a rectangular garden next to a house. He had 110 ft. of fencing to put along the three sides of the garden and wanted the area to be 1500 sq. ft. What were the dimensions of the garden?

2. A circular lily pond is surrounded by a walk 4 ft. wide. The area of the walk is $\frac{13}{36}$ of the area of the pond. Find the radius of the pond.

3. The sum of a number and its reciprocal is $\frac{10}{3}$. Find the number. (The reciprocal of x is 1 divided by x.)

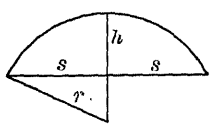
4. The sum of a number and six times its reciprocal equals $5\frac{1}{2}$. Find the number.

5. Find two consecutive integers such that the smaller plus the square of the larger equals 19.

6. Find two consecutive odd integers such that their product plus their sum equals 23.

7. A rectangular piece of tin is 15 in. by 20 in. How wide a strip must be cut from all four sides to make the remaining piece half as large as the original piece?

8. In building a circular arch the radius, r, is found by solving the equation $h^2 - 2hr + s^2 = 0$, in which h = height of the arch, and s = half the span of the arch.



If r is 15 ft., and the total span, $2s$, is 18 ft., find the value of h.

Can both roots be used in the problem?

9. If a club has n members and 2 members are to be selected as officers, the number of ways, N, of selecting these two is given by the formula $N = \frac{1}{2}n(n-1)$. If $N = 435$, how many members has the club?

10. If a polygon has n sides, the total number of lines, N, that can be drawn joining any two corners is given by the formula $N = \frac{1}{2}n(n-3)$. If $N = 14$, how many sides has the polygon?

190. The solution of literal quadratics by the formula is an excellent review and test of the pupil's ability to handle fractions, parentheses, radicals, etc.

EXAMPLE. Solve for x: $\quad 3x^2 + 5x - 2kx = 2 + kx^2$

First rewrite the equation so that the coefficients of x^2 and x and the terms without x can be seen easily. Thus:

$$(3 - k)x^2 + (5 - 2k)x - 2 = 0$$

Hence $\quad a = 3 - k, \quad b = 5 - 2k, \quad c = -2$

Then $\quad x = \dfrac{-(5 - 2k) \pm \sqrt{(5 - 2k)^2 + 8(3 - k)}}{2(3 - k)}$

The expression under the radical sign reduces to

$$25 - 20k + 4k^2 + 24 - 8k = 49 - 28k + 4k^2 = (7 - 2k)^2$$

Then $\quad x = \dfrac{-(5 - 2k) \pm (7 - 2k)}{2(3 - k)}$

One root is $\quad x = \dfrac{-5 + 2k + 7 - 2k}{2(3 - k)} = \dfrac{2}{2(3 - k)} = \dfrac{1}{3 - k}$

The other root is $x = \dfrac{-5 + 2k - 7 + 2k}{2(3 - k)} = \dfrac{4k - 12}{2(3 - k)} = -2$

Check the work by finding the sum and the product of the roots.

191. EXERCISES

Using the formula, solve for x and check:

1. $gx^2 - 2gx + 3x = 6$
2. $hx^2 - 3hx + 3 = 7x - 2x^2$
3. $2gx^2 - g^2x = 6x - 3g$
4. $mx^2 - n(m + 1)x + n^2 = 0$
5. $rsx^2 + rs = x(r^2 + s^2)$
6. $2(g - 1)x^2 + 2(g + 1) = x(g^2 + 3)$
7. $x^2 - 2hx - kx + h^2 + hk = 2k^2$
8. $x^2 - 2x + kx + 1 = kx^2$
9. $hx^2 + 3x + 2 = 2x^2 + 2hx$
10. $hkx^2 + hk = x(h + k)\sqrt{hk}$
11. $2x^2 - x\sqrt{2r} + \sqrt{rs} = x\sqrt{2s}$
12. $t^2x^2 + rs = rtx + stx$

192. Equations Reducible to the Quadratic Form. Many equations of higher degree can be changed by various substitutions to quadratic equations and thereby solved.

EXAMPLES

1. $x^6 - 7x^3 - 8 = 0$

Here one exponent, 6, is double the other exponent, 3. Substitute y for x^3. Then $y^2 - 7y - 8 = 0$. After finding that $y = 8$ and $y = -1$, find x by solving the equations: $x^3 = 8$ and $x^3 = -1$.

There are three cube roots of 8 (see page 147, ex. 28) and of -1, but only the real root need be found here.

2. $(x^2 - 3x)^2 - 2(x^2 - 3x) - 8 = 0$

The repetition of the quantity $(x^2 - 3x)$ suggests that y be substituted for $x^2 - 3x$. The equation then becomes $y^2 - 2y - 8 = 0$. After finding that $y = -2$, and $y = 4$, find x by solving the two new equations:

$$x^2 - 3x = -2 \quad \text{and} \quad x^2 - 3x = 4$$

Could you recognize the proper substitution if the same equation were written: $(x^2 - 3x)^2 - 2x^2 + 6x - 8 = 0$? if it were written: $x^4 - 6x^3 + 9x^2 - 2(x^2 - 3x) = 8$?

3. $\left(x - \frac{8}{x}\right)^2 + 9\left(x - \frac{8}{x}\right) + 14 = 0$

Substitute y for $\left(x - \frac{8}{x}\right)$

4. $x^2 - 3x + 4\sqrt{x^2 - 3x} - 12 = 0$

Substitute y for $\sqrt{x^2 - 3x}$. The equation then becomes $y^2 + 4y - 12 = 0$, whose roots are 2 and -6. Hence

$$\sqrt{x^2 - 3x} = 2, \quad \text{and} \quad \sqrt{x^2 - 3x} = -6$$

The first equation is then changed to $x^2 - 3x = 4$.

The second equation must be discarded, as it is impossible for a positive radical to equal -6 (see page 122).

193. EXERCISES

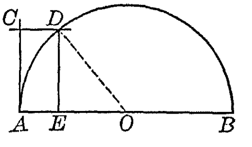
Find the real roots of the following equations:

1. $x^6 - 9x^3 + 8 = 0$
2. $8x^6 - 35x^3 + 27 = 0$
3. $x^4 - 13x^2 + 36 = 0$
4. $4x^4 - 41x^2 + 100 = 0$
5. $x - 5x^{\frac{1}{2}} + 6 = 0$
6. $2x - 7x^{\frac{1}{2}} + 3 = 0$
7. $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$
8. $10x^{\frac{2}{3}} - x^{\frac{1}{3}} - 3 = 0$
9. $5x^{-2} - 17x^{-1} + 6 = 0$
10. $4x^{-4} - 5x^{-2} + 1 = 0$
11. $(x^2 - 3x)^2 - 14(x^2 - 3x) + 40 = 0$
12. $(x^2 + 2x)^2 - 11(x^2 + 2x) + 24 = 0$
13. $x^4 + 10x^3 + 25x^2 + 10(x^2 + 5x) + 24 = 0$
14. $(2x^2 - 5x)^2 + 10x^2 - 25x + 6 = 0$
15. $x^4 - 4x^3 + 4x^2 - 8x^2 + 16x + 15 = 0$
16. $\left(x - \frac{6}{x}\right)^2 + 4\left(x - \frac{6}{x}\right) - 5 = 0$
17. $6\left(x + \frac{1}{x}\right)^2 - 35\left(x + \frac{1}{x}\right) + 50 = 0$
18. $3\left(x + \frac{2}{x}\right)^2 - 20\left(x + \frac{2}{x}\right) + 33 = 0$
19. $2\left(x - \frac{3}{x}\right)^2 + 3\left(x - \frac{3}{x}\right) - 2 = 0$
20. $\left(x - \frac{6}{x}\right)^2 - 6\left(x - \frac{6}{x}\right) + 5 = 0$
21. $x^2 - 3x + 2\sqrt{x^2 - 3x} - 8 = 0$
22. $x^2 + 3x - 5\sqrt{x^2 + 3x} + 6 = 0$
23. $x^2 - 8x - 4\sqrt{x^2 - 8x} + 3 = 0$
24. $x^2 - 5x + 10 - \sqrt{x^2 - 5x + 10} - 12 = 0$
25. $x^2 + 2x - 5\sqrt{x^2 + 2x + 6} + 12 = 0$
26. $(x^2 - 6x + 3)^2 + 7x^2 - 42x + 31 = 0$
27. $(x^2 - 8x + 14)^2 - x^2 + 8x - 16 = 0$

194. Historical Note. Greek mathematicians as early as Euclid, who lived in Alexandria about 300 years B.C., solved quadratic equations by geometric methods. The pupil may have learned in geometry how the solution of the equation $x^2 + ax - a^2 = 0$ is equivalent to dividing a line of length a into extreme and mean ratio, the greater segment of the line being one root of the equation. The figure below shows how the equation $x^2 - px + q = 0$ can be solved geometrically. Since the lines AE and EB are respectively equal to $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$ and $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ their lengths are the roots of the equation.

CONSTRUCTION:

Draw a semicircle with $AO = \frac{1}{2}p$.
Draw $AC \perp AB$, and make $AC = \sqrt{q}$.
Draw $CD \parallel AB$ and $DE \perp AB$.



On page 56 it was stated that the number of roots of an equation is the same as the degree of the equation. On solving $x^2 - 6x + 9 = 0$, we find that 3 is the only number that satisfies the equation; nevertheless, we say that the equation has two roots because the quantity $x^2 - 6x + 9$ has two factors, the roots in this case being equal.

For several centuries following Euclid, mathematicians were satisfied to find one root to a quadratic equation, and they regarded negative numbers as unsatisfactory. Now we know that even the square roots of negative numbers can be satisfactorily interpreted as roots.

CHAPTER XII

THEORY AND GRAPHS OF QUADRATIC EQUATIONS

195. Theory of Equations. The roots of an equation naturally depend on the coefficients in the equation. If the numbers a, b, and c in $ax^2 + bx + c = 0$ are changed, the roots will be changed. Hence one of the interesting parts of mathematics is the study of the relation between the coefficients and the roots of an equation.

For the quadratic equation $ax^2 + bx + c = 0$ we know that:

$$-\frac{b}{a} = r_1 + r_2 \qquad \frac{c}{a} = r_1 r_2 \quad \text{(See page 158.)}$$

Since we can make $a = 1$ by dividing the equation by the coefficient of x^2, these relations can be written as:

$$b = -(r_1 + r_2) \qquad c = r_1 r_2 \qquad (\text{when } a = 1)$$

For the cubic equation $x^3 + bx^2 + cx + d = 0$ with the three roots r_1, r_2, and r_3 it can be proved that

$$b = -(r_1 + r_2 + r_3) \qquad c = r_1 r_2 + r_1 r_3 + r_2 r_3 \qquad d = -\, r_1 r_2 r_3$$

That is, b is the negative of the sum of the roots, c is the sum of the products of the roots taken two at a time, and d is the negative of the product of the roots.

That part of mathematics which studies the relations between the coefficients of an equation and its roots, and also many other useful theorems about equations is called the *Theory of Equations*.

In this chapter we shall deal only with the quadratic equation. We shall see that we can determine the nature of its roots (that is, whether the roots are real or imaginary, etc.) without solving the equation.

196. Character of the Roots of a Quadratic Equation. The roots of the equation $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The radicand $b^2 - 4ac$ is called the *discriminant* because its value decides whether the roots are (1) equal or unequal, (2) real or complex, (3) rational or irrational.

I. *If* $b^2 - 4ac$ *is zero, the roots are equal and real.*

Thus, in $4x^2 - 12x + 9 = 0$, we have $a = 4$, $b = -12$, $c = 9$. Since $b^2 - 4ac = 144 - 144 = 0$, the roots are equal. The usual solution gives as the roots: $\frac{12 + \sqrt{0}}{8}$ and $\frac{12 - \sqrt{0}}{8}$, which both reduce to $1\frac{1}{2}$. It is true that there is only *one* number that satisfies this equation, but, as explained on page 164, it is customary to say that there are *two equal* roots.

II. *If* $b^2 - 4ac$ *is positive, the roots are unequal and real.*

The roots are real because the discriminant is positive. They are unequal because in one root the radical has a positive sign and in the other root a negative sign.

Thus, in $3x^2 - 7x - 20 = 0$, we have $a = 3$, $b = -7$, $c = -20$; and $b^2 - 4ac = 49 + 240 = 289$. The roots are 4 and $-\frac{5}{3}$.

III. *If* $b^2 - 4ac$ *is negative, the roots are unequal and are complex numbers.*

The roots are complex numbers because the square root of a negative number is an imaginary number.

Thus, in $3x^2 - 2x + 1 = 0$, we have $a = 3$, $b = -2$, $c = 1$. Since $b^2 - 4ac = 4 - 12 = -8$, the roots are complex numbers. In fact, the roots are $\frac{2 + \sqrt{-8}}{6}$ and $\frac{2 - \sqrt{-8}}{6}$.

IV. *If* $b^2 - 4ac$ *is an exact square (or zero), the roots are rational; otherwise they are irrational.*

Thus, in $x^2 - 5x + 4 = 0$, we have $b^2 - 4ac = 9$; hence the roots are rational. But in $x^2 - 5x - 3 = 0$, we have $b^2 - 4ac = 37$ and hence the roots involve the irrational number $\sqrt{37}$.

197. EXERCISES

Find the discriminants of equations 1 to 10, and then state the character of their roots:

1. $x^2 - 2x - 14 = 0$
2. $x^2 - 10x + 25 = 0$
3. $9x^2 + 3x - 2 = 0$
4. $6x^2 + 7x - 20 = 0$
5. $6x^2 - 7x - 20 = 0$
6. $9x^2 - 24x + 16 = 0$
7. $x^2 + x + 1 = 0$
8. $x^2 + x + \frac{1}{4} = 0$
9. $x^2 - x - 2 = 0$
10. $x^2 - x + 3 = 0$

11. For what value of k will the roots of the equation $x^2 - 7x + 3k = 0$ be equal?

SUGGESTION. Here $a = 1$, $b = -7$, $c = 3k$, and $b^2 - 4ac = 49 - 12k$. Hence if $49 - 12k = 0$, the roots will be equal. $k = ?$ Prove that your value is correct by finding the roots.

As in ex. 11, find the values of k for which the two roots of each of the following equations are equal:

12. $4x^2 + kx + 9 = 0$
13. $kx^2 - 6x + 9 = 0$
14. $2x^2 - 3x + k = 0$
15. $kx^2 + 4x + 1 = 0$
16. $2x^2 + kx + 3 = 0$
17. $(k+5)x^2 - 8x + 16 = 0$
18. $x^2 - (k+2)x + 4 = 0$
19. $kx^2 - 3kx + 9 = 0$
20. $x^2 - (3-k)x + 25 = 0$
21. $x^2 + (3-k)x + k = 0$

22. If $x = 3$ is one root of the equation $x^2 - x + k = 4$, find k. (Substitute 3 for x in the equation.)

23. In the equation $4x^2 - 24x + c = 0$, find the value of c if one root is 5 more than the other root.

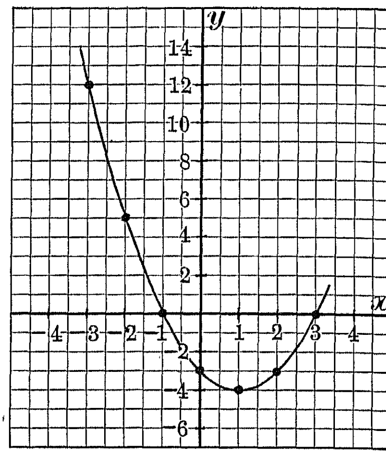
SUGGESTION. Call the roots r and $r + 5$. The sum of the roots is $-\frac{b}{a}$; that is, $r + (r+5) = \frac{24}{4}$, or 6. The product of the roots is $\frac{c}{a}$; that is, $r(r+5) = \frac{c}{4}$. Solve these equations for r and c.

24. In the equation $9x^2 - (k+1)x + 8 = 0$, find the value of k if one root is twice the other root.

25. Find k if one root of $x^2 + 4kx - 2 = 0$ is 3 more than the other root.

198. Graphs of Quadratic Equations. To draw the graph of $y = x^2 - 2x - 3$, we first find the values of y when x is given some values like $-3, -2, -1, 0, 1, 2, 3$.

x	y
-3	12
-2	5
-1	0
0	-3
1	-4
2	-3
3	0



After drawing part of the curve, we see that it may be advisable to extend the curve to the right or left. We find the values of y for $x = 4$, $x = 5$, and plot these points.

The values of x at the points where the curve crosses the x axis are -1, and 3. The values of y at these points are 0 in all cases. Moreover, if we substitute 0 for y in the given equation, and solve $0 = x^2 - 2x - 3$ we find that $x = -1$, and $x = 3$ are the roots.

This work illustrates the following important fact:

The roots of the equation $ax^2 + bx + c = 0$ *are the values of* x *at the points where the curve* $y = ax^2 + bx + c$ *crosses the* x *axis.*

We have thus seen how (1) the equation may be used to find exactly where the curve crosses the axis if the graph does not clearly show it, and (2) the graph may be used to find the roots of the equation.

The graphic method shown here for solving quadratic equations may be applied to all kinds of equations.

199. EXERCISES

Draw the graphs for the following equations. Use values of x from -3 to $+3$ at first.

1. $y = x^2 - 2x + 3$
2. $y = 2x^2 + x - 10$
3. $y = x^2 + x - 10$
4. $y = 6 + x - x^2$
5. $y = 4 + 3x - x^2$
6. $4y = x^2 - 16$
7. $3y = x^2 - 15$
8. $2y = x^2 - 9$
9. $2x + 3 = x^2 - y$
10. $x + y = x^2 - 6$

11. On the same set of axes draw the graphs for: $y = x^2 - 2x - 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x + 5$.

Notice that the only change in the equations is in the term that does not contain x. What is the difference in the graphs of the three equations? How can we tell from a graph whether an equation has two real roots or two equal roots or has complex numbers for the roots?

12. Draw the graph of $y = x^2 - 2x$ and on the same axes the graph for $y = -x^2 + 2x$. How do they differ?

13. Draw the graph of $y = 2x^3 - x^2 - 8x + 3$. This is not a quadratic equation, and the curve has more turns in it than any of the previous curves. The method of drawing the graph is the same no matter what the degree of the equation is; that is, the values of y are found for some values of x, and these points are plotted.

At what points, approximately, does the curve cross the x axis? What are the roots of $2x^3 - x^2 - 8x + 3 = 0$?

As in ex. 13, solve the following equations by drawing the graphs and noting where the curve crosses the x axis:

14. $4x^3 - 40x = 0$

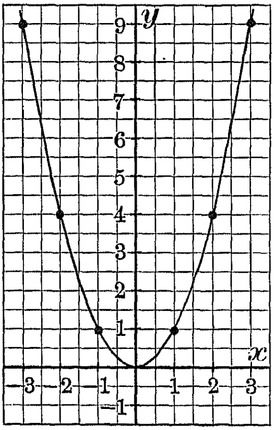
15. $2x^3 + 3x^2 - 20x - 21 = 0$

16. $x^3 - 2x^2 - 9x + 15 = 0$

17. $3x^3 + 15x^2 + 18x = 0$

18. $x^3 - 2x^2 - 7x - 4 = 0$

200. Graphs of Some Fundamental Equations. There are certain equations that occur so frequently in various fields of mathematics that a knowledge of their graphs is important. They are:

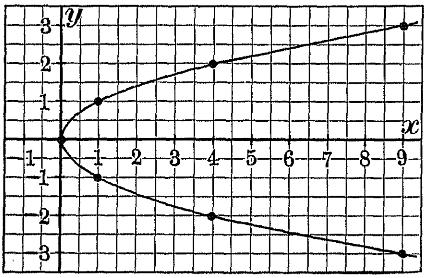


1. $y = x^2$

This curve is called a *parabola.* The values of y are all positive whether x is positive or negative.

The equation $A = s^2$, which gives the area of a square whose side is s, is of this kind.

When the equation is $y = kx^2$ the curve becomes broader or narrower according to the value of the number k.



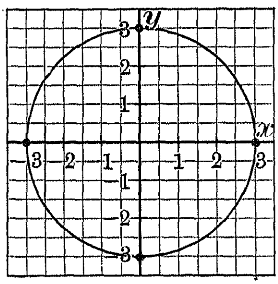
2. $y^2 = x$, or

$$y = \pm\sqrt{x}$$

The equations

$$t^2 = \pi\,\frac{l}{32.2}$$

and $m^2 = \frac{3}{2}h$ (page **192**, ex. 36) are of this kind.



3. $x^2 + y^2 = r^2$,

or $$y = \pm\sqrt{r^2 - x^2}$$

(In the figure, $r = 3$.)

The curve is a circle if the unit distance, from 0 to 1, is the same on both axes. The center of the circle is at the origin, and the radius of the circle is the constant r.

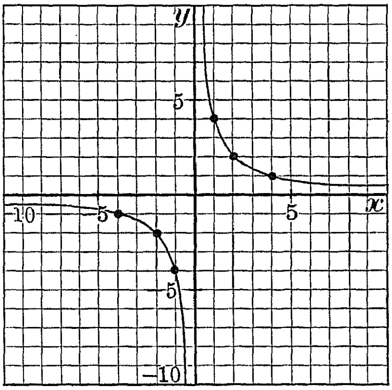
4. $xy = k$, or $y = \dfrac{k}{x}$

(In the figure, $k = 4$.)

The curve is called a *hyperbola,* and consists of two distinct branches.

There is no value of y for $x = 0$ because we cannot divide by zero.

The two branches each come closer and closer to the axes when the branches are prolonged.

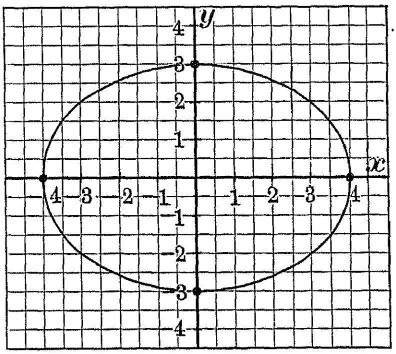


5. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(In the figure, $a = 4$ and $b = 3$; the equation is then $9x^2 + 16y^2 = 144$.)

The curve is an *ellipse.*

The paths described by the planets and some of the comets moving around the sun are ellipses.



201. EXERCISES

1. On one set of axes draw the curves for:

$y = 3x^2, \quad y = x^2, \quad y = \frac{1}{3}x^2, \quad y = -\frac{1}{3}x^2, \quad y = -x^2$

2. Show that the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be written as $y = \pm \dfrac{b}{a}\sqrt{a^2 - x^2}.$

Draw the curve when $a = 5$, $b = 4$.

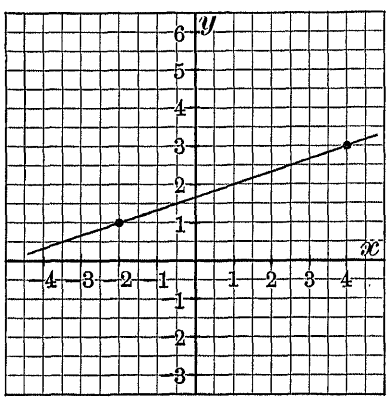
What is the nature of the curve when $a = b$?

(Supplementary Topics, Pages 172 to 174)

202. Finding the Equation of a Curve Passing through Given Points. We have learned how to draw the curve represented by any equation. We shall see now how to find the equation of a curve when we know some of the points on it. The scientist must frequently do work of this kind, using data obtained in experiments.

Problem. Find the equation of the straight line that passes through the points $x = -2$, $y = 1$ and $x = 4$, $y = 3$.

If the equation represents a straight line, it must be of the first degree in x and y, like $y = 2x + 3$ or $y = 4x + 5$. From the information in the problem we determine what the true coefficients are.



Let the equation of the line be

$$y = mx + b \qquad (1)$$

wherein m and b represent the numbers whose values we must find.

Since the line passes through the point for which $x = -2$, $y = 1$, equation (1) must be true if we substitute these values in it.

Hence $$1 = -2m + b \qquad (2)$$

Likewise if we substitute $x = 4$, $y = 3$ in equation (1),

$$3 = 4m + b \qquad (3)$$

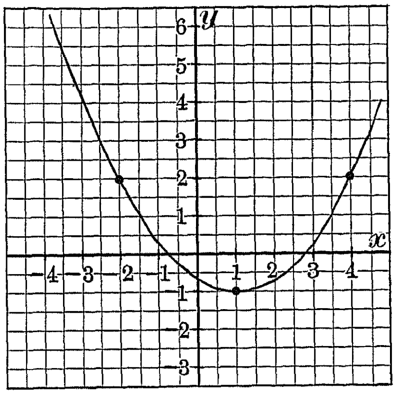
We thus have two equations, (2) and (3), from which we can find the values of m and b. The solution is $m = \frac{1}{3}$, $b = \frac{5}{3}$. Writing these values of m and b in equation (1), we have

$$y = \tfrac{1}{3}x + \tfrac{5}{3}$$

This is the equation of the line through the given points.

Problem. Find the equation of the parabola that passes through the points $(-2, 2)$, $(1, -1)$, and $(4, 2)$.

The point $(-2, 2)$ means the point for which $x = -2$, $y = 2$.



From methods that cannot be explained here, it is known that the equation of a parabola is

$$y = ax^2 + bx + c \quad (1)$$

where a, b, and c are the constants we are to find.

As in the previous problem, we substitute the given values of x and y in (1) and get a set of three equations to solve:

$$2 = 4a - 2b + c, \quad -1 = a + b + c, \quad 2 = 16a + 4b + c$$

The solution of this set is: $a = \frac{1}{3}$, $b = -\frac{2}{3}$, $c = -\frac{2}{3}$. Hence the equation of the parabola is $y = \frac{1}{3}x^2 - \frac{2}{3}x - \frac{2}{3}$.

203. EXERCISES

Find the equation of the straight line that passes through:

1. $(-2, -4)$ and $(1, 5)$
2. $(2, 1)$ and $(-2, 9)$
3. $(-2, 1)$ and $(4, 7)$
4. $(4, 5)$ and $(2, 1)$
5. $(1, 2)$ and $(-5, 8)$
6. $(4, \frac{2}{3})$ and $(-2, -\frac{4}{3})$
7. $(1, 2)$ and $(4, 1)$
8. $(7, 2\frac{1}{2})$ and $(5, 2)$
9. $(4, -2)$ and $(3, -2\frac{1}{4})$
10. $(2, \frac{2}{3})$ and $(1, \frac{1}{6})$

Find the equation of the parabola through the points:

11. $(1, -1)$, $(-2, 11)$, and $(3, 1)$
12. $(-1, 5)$, $(0, 2)$, and $(2, 8)$
13. $(2, 1)$, $(4, 9)$, and $(-1, 4)$
14. $(2, 7)$, $(-3, 12)$, and $(-1, -2)$
15. $(1, 0)$, $(2, 4)$, and $(-1, 10)$

204. Forming an Equation with Given Roots. By reversing the four steps shown on page 57, § 56, we can form an equation that shall have certain roots. Thus:

If the desired roots are $x = 3$ and $x = -2$, then the preceding step was $x - 3 = 0$ and $x + 2 = 0$ and the step preceding this was $(x - 3)(x + 2) = 0$. Hence the required equation is $x^2 - x - 6 = 0$.

Again, the equation whose roots are $x = \frac{2}{3}$ and $x = \frac{1}{4}$ is $(x - \frac{2}{3})(x - \frac{1}{4}) = 0$, or $x^2 - \frac{11}{12}x + \frac{1}{6} = 0$, which might be written as $12x^2 - 11x + 2 = 0$.

Another method of finding the equation is to assume that it is $x^2 + px + q = 0$, and then find the numbers p and q by means of the rules on page 158. Thus, if the roots are to be $h + k$ and $h - 2k$, then

$$-p = (h + k) + (h - 2k) \quad \text{and} \quad q = (h + k)(h - 2k);$$

that is, $p = -2h + k$ and $q = h^2 - hk - 2k^2$. When these values of p and q are substituted in $x^2 + px + q = 0$ we get
$$x^2 - 2hx + kx + h^2 - hk - 2k^2 = 0.$$

205. **EXERCISES**

Find the quadratic equations whose roots are:

1. -3 and 5
2. $\frac{2}{3}$ and $\frac{5}{6}$
3. h and $2h$
4. $5h$ and $-\frac{1}{2}h$
5. 3.7 and 5.4
6. $h - k$ and $h + 3k$
7. $s + t$ and $s - t$
8. $\dfrac{h + k}{2}$ and $\dfrac{2}{h + k}$
9. $2 + \sqrt{3}$ and $2 - \sqrt{3}$
10. $\frac{1}{2}(3 + \sqrt{17})$ and $\frac{1}{2}(3 - \sqrt{17})$
11. $2 + \sqrt{-3}$ and $2 - \sqrt{-3}$

In ex. 12 to 15, $i = \sqrt{-1}$.

12. $5 + 3i$ and $5 - 3i$
13. $r + is$ and $r - is$
14. $(r + s) + i(r - s)$ and $(r + s) - i(r - s)$
15. $r + s + mi$ and $r + s - mi$

CHAPTER XIII

SETS INVOLVING QUADRATIC EQUATIONS

206. Problem. A party of boys bought a canoe for \$60, dividing the cost equally. If the number of boys had been 4 less, the cost per boy would have been \$1.25 greater. How many boys were there in the party?

The words "if the number of boys had been 4 less" suggest that we use b and $b - 4$ to represent the number of boys. The words "the cost per boy would have been \$1.25 greater" suggest that we use c and $c + 1.25$ to represent the cost per boy. Hence we write:

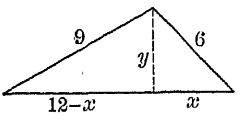
	Number of boys	Cost per boy	Total cost
(1)	b	c	60
(2)	$b - 4$	$c + 1.25$	60

We now see that the two equations are:

$$\text{from line (1)}: \quad bc = 60$$
$$\text{from line (2)}: \quad (b-4)(c+1.25) = 60$$

Before solving this set of equations (see page 177), let us consider the following problem:

Problem. The sides of a triangle are 9 in., 12 in., and 6 in. Find the altitude on the side 12 in.



From the figure we see that the equations are:

$$(12 - x)^2 + y^2 = 81$$
$$x^2 + y^2 = 36$$

Unlike the sets of equations in Chapter VII, these equations are of the second degree in the unknown numbers. In this chapter we shall study such sets of quadratic equations.

207. One Linear, One Quadratic, Equation.

If one of the equations is of the first degree in x and y, solve that equation either for x or for y, and substitute the result in the other equation.

EXAMPLE. Solve $\begin{cases} 3x + 2y = 4 & (1) \\ x^2 + y^2 = 5 & (2) \end{cases}$

Solve equation (1) for y: $2y = 4 - 3x$, or $y = \dfrac{4 - 3x}{2}$

Substitute this value of y in equation (2):

$$x^2 + \left(\frac{4 - 3x}{2}\right)^2 = 5, \text{ or } x^2 + \frac{16 - 24x + 9x^2}{4} = 5$$

Then $\quad 4x^2 + 16 - 24x + 9x^2 = 20$

or $\quad 13x^2 - 24x - 4 = 0.$ Hence $x = 2$ or $-\frac{2}{13}$.

Substitute the value of x in equation (1):

If $x = 2$, then $6 + 2y = 4$, or $y = -1$.

If $x = -\frac{2}{13}$, then $-\frac{6}{13} + 2y = 4$, or $y = \frac{29}{13}$.

Check the solutions by substituting them in (1) and (2).

The answers to a set of equations should be so written that we can see which values of x and y correspond. Thus:

$\boxed{x = 2, y = -1}$ $\quad$ $\boxed{x = -\frac{2}{13}, y = \frac{29}{13}}$

208. EXERCISES

Solve the following sets of equations, and check:

1. $\begin{cases} x + y = 3 \\ 2x^2 + y^2 = 9 \end{cases}$

2. $\begin{cases} 2x + y = 2 \\ 3x^2 + xy = 15 \end{cases}$

3. $\begin{cases} 3y + x = 10 \\ 3y = x^2 + 8 \end{cases}$

4. $\begin{cases} y + 3x = 9 \\ y^2 + 9x^2 = 45 \end{cases}$

5. $\begin{cases} 2y + 9x = 7 \\ 9x^2 + 7 = 2y^2 \end{cases}$

6. $\begin{cases} 3x + 4y = 18 \\ x^2 + 2y = 10 \end{cases}$

7. $\begin{cases} 2xy + 1 = 0 \\ y + 4x = 1 \end{cases}$

8. $\begin{cases} 2x + 3y = 6 \\ 4xy = 2x - 15 \end{cases}$

209. If neither equation of a set is linear, a linear equation may sometimes be found by adding or subtracting the given equations.

For example, in the first problem on page 175 the equations

are $$bc = 60 \qquad (1)$$

and $$bc + 1.25\,b - 4\,c - 5 = 60 \qquad (2)$$

If the second equation is subtracted from the first, then

$$-1.25\,b + 4\,c + 5 = 0$$

Solve this equation for c and substitute the result in (1).

Again, in the second problem on page 175, the equations

are $$144 - 24\,x + x^2 + y^2 = 81 \qquad (1)$$

and $$x^2 + y^2 = 36 \qquad (2)$$

If the second equation is subtracted from the first, then

$$144 - 24\,x = 45$$

Solve this equation for x, and then find y.

210. EXERCISES

Solve the following sets of equations, and check:

1. $\begin{cases} xy = 12 \\ (x-1)(y-2) = 4 \end{cases}$

2. $\begin{cases} xy = -15 \\ (x+1)(y+6) = 4 \end{cases}$

3. $\begin{cases} rt + 3\,t = 5 \\ (r+3)(t+1) = 10 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 13 \\ (1+x)^2 + y^2 = 18 \end{cases}$

5. $\begin{cases} u^2 - v^2 = 16 \\ (u-1)^2 + v^2 = 25 \end{cases}$

6. $\begin{cases} 4\,r^2 + s^2 = 8 \\ (2\,r-1)^2 + s^2 = 5 \end{cases}$

7. $\begin{cases} uv = 3 \\ (u+1)(v-2) = 2 \end{cases}$

8. $\begin{cases} xy = -6 \\ x(y+2) - 3\,y = 7 \end{cases}$

9. $\begin{cases} xy + y = 12 \\ (3-y)(1+x) = 3\,y \end{cases}$

10. $\begin{cases} rs + r^2 + 1 = 0 \\ (r+s)(r-2) = 1 \end{cases}$

11. $\begin{cases} 2\,x + 3\,y = 6 \\ y(4\,x + 3) + 9 = 0 \end{cases}$

12. $\begin{cases} 6\,uv = 1 \\ (3\,u+1)(2\,v+3) = 8 \end{cases}$

13. $\begin{cases} 2\,xy + 1 = 0 \\ 2(x+1)(y+4) = 9 \end{cases}$

14. $\begin{cases} a^2 + 3\,b^2 = 7 \\ (a-1)^2 + 3\,b^2 = 12 \end{cases}$

211. **PROBLEMS**

1. The sum of the squares of two numbers is 13, and the smaller is 10 less than 4 times the larger. Find the numbers. (We can easily guess that one pair of numbers is 3 and 2; the equations will show that there is also another pair of satisfactory numbers.)

2. In placing some telephone poles it was found that by increasing the distance, d, between them 10 ft., 4 poles less per mile would be used. What is the distance between the poles and the number, n, of poles per mile?

3. The cost of a trip for a number of men was \$72 and was to be shared equally. Because 6 men failed to appear, however, the cost per man was increased \$2. What was the original number of men? (Use two letters, m and c.)

4. If a man's daily wage were \$3 less, he would have to work 8 days more to earn \$144. What is his daily wage?

5. If a man's daily wage were \$3 less, he would have to work 6 days more to make \$140. What is his daily wage?

6. A man traveled 210 mi. If he had gone 5 mi. an hour faster, he would have made the trip in 7 hr. less time. Find his rate in miles per hour.

7. Some books were purchased for \$150. If the price per book had been \$.10 less, 50 more books could have been bought for the same price. What was the price per book?

8. The sides of a triangle are 10 in., 17 in., and 21 in. Find the length of the altitude to the side 21 in.

9. The sides of a triangle are 11 in., 25 in., and 30 in. Find the altitude to the side 11 in. (Draw a figure to scale.)

10. Find the equations for the following problem:

The circumference of a wheel is c. By how much must the circumference be increased so that the wheel will make n fewer revolutions in going a distance d? (Let the increase be x; let the number of revolutions be y and $y - n$.)

212. If a linear equation cannot be obtained as shown on page 177, try one of the following methods.

EXAMPLE 1. Solve $\begin{cases} 3x^2 - 8y^2 = 4 & (1) \\ 2x^2 + 5y^2 = 13 & (2) \end{cases}$

These equations contain only terms like x^2 and y^2. They are called *linear in x^2 and y^2*, because if letters are substituted, as u for x^2 and v for y^2, the new equations will be linear in u and v; namely,

$$3u - 8v = 4 \quad \text{and} \quad 2u + 5v = 13$$

You need not actually make the substitution. Multiply (1) by 5 and (2) by 8, and add. Then $31x^2 = 124$, or $x^2 = 4$. Hence $x = \pm 2$. Then we can find y as we usually do.

EXAMPLE 2. Solve $\begin{cases} x^2 + y^2 = 13 & (1) \\ xy = 6 & (2) \end{cases}$

Solve (2) for y, getting $y = \frac{6}{x}$. Then $x^2 + \frac{36}{x^2} = 13$. Hence $x^4 - 13x^2 + 36 = 0$, which is quadratic in form. (See page 162.)

213. EXERCISES

Solve the following sets of equations:

1. $\begin{cases} 8x^2 - y^2 = 5 \\ x^2 + 5y^2 = 16 \end{cases}$

2. $\begin{cases} x^2 + 6y^2 = 15 \\ 2x^2 - 9y^2 = 9 \end{cases}$

3. $\begin{cases} 3x^2 + 4y^2 = 31 \\ 5x^2 - 6y^2 = 1 \end{cases}$

4. $\begin{cases} 4x^2 - y^2 = 3\frac{3}{4} \\ 6x^2 + 3y^2 = 6\frac{3}{4} \end{cases}$

5. $\begin{cases} xy + 2 = 0 \\ 4x^2 + y^2 = 17 \end{cases}$

6. $\begin{cases} 9x^2 + 8y^2 = 3 \\ 6xy - 1 = 0 \end{cases}$

7. $\begin{cases} 5x^2 + y^2 = 6 \\ 2(x + \frac{1}{2})^2 = y^2 - \frac{1}{2} \end{cases}$

In ex. 7, add the two equations to eliminate y^2.

8. $\begin{cases} 5x^2 + 2xy = 6 \\ 10x - 3xy = 1 \end{cases}$

In ex. 8, multiply the first equation by 3, the second by 2, and then add.

9. $\begin{cases} x^2 + 4xy = 8 \\ x - xy = 1 \end{cases}$

10. $\begin{cases} 2xy + y^2 = 3 \\ 5xy - y = 6 \end{cases}$

11. $\begin{cases} 4x^2 + y^2 = 52 \\ (2x - 1)^2 = y^2 - 11 \end{cases}$

12. $\begin{cases} xy = \sqrt{18} \\ y^2 + 4x^2 = 17 \end{cases}$

214. EXERCISES FOR DRILL

Solve the following sets of equations:

1. $\begin{cases} 4x^2 + y^2 = 5 \\ xy + 1 = 0 \end{cases}$

2. $\begin{cases} 2x^2 + 9y^2 = 9 \\ 3xy + 2 = 0 \end{cases}$

3. $\begin{cases} x^2 + y^2 = 5 \\ x^2 - y^2 = 3 \end{cases}$

4. $\begin{cases} x^2 = (10 - y)(10 + y) \\ x^2 - y^2 = 28 \end{cases}$

5. $\begin{cases} y^2 = (25 + x)(25 - x) \\ 2x^2 - y^2 = 50 \end{cases}$

6. $\begin{cases} x^2 = (5 + y)(5 - y) \\ xy = 12 \end{cases}$

7. $\begin{cases} y^2 = (\frac{5}{12} - x)(\frac{5}{12} + x) \\ 12xy = 1 \end{cases}$

8. $\begin{cases} y^2 = (13 - x)(13 + x) \\ xy = 60 \end{cases}$

9. $\begin{cases} 4x^2 + 9y^2 = 2 \\ 6xy = 1 \end{cases}$

10. $\begin{cases} y^2 + 8x^2 = 11 \\ 2xy = 3 \end{cases}$

11. $\begin{cases} 2x + 3y = 8 \\ x^2 + y^2 = 5 \end{cases}$

12. $\begin{cases} y^2 + 2x = 17 \\ 2y - x = 2 \end{cases}$

13. $\begin{cases} 2x - 15y = 18 \\ xy = 48 \end{cases}$

14. $\begin{cases} xy = 12 \\ 3x + 2y = 18 \end{cases}$

15. $\begin{cases} y^2 - x^2 = \frac{5}{6}xy \\ 3x + 2y = 2 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 13 \\ (x - 1)^2 + y^2 = 10 \end{cases}$

17. $\begin{cases} 3x + 4y = 5 \\ 2xy - 6x = -3 \end{cases}$

18. $\begin{cases} 4x^2 + y^2 = 16 \\ (2x - 1)^2 = 9 + y^2 \end{cases}$

19. $\begin{cases} x^2 + 4y^2 = 8 \\ x^2 + (1 + y)^2 = 8 \end{cases}$

20. $\begin{cases} y^2 = 2 - 4x^2 \\ y^2 + (1 - 2x)^2 = 1 \end{cases}$

Solve for x and y:

21. $\begin{cases} ax + by = 2ab \\ xy = ab \end{cases}$

22. $\begin{cases} bx + ay = a^2 + b^2 \\ xy = a^2 - b^2 \end{cases}$

23. $\begin{cases} bx - ay = ab^2 \\ xy = ab(1 - b) \end{cases}$

24. $\begin{cases} 3bx + 2ay = 12ab \\ xy = 6ab \end{cases}$

25. $\begin{cases} ax - y = ab \\ x^2 - y = 2ab + b^2 \end{cases}$

26. $\begin{cases} ax - y = a \\ xy = a^2(x - 1) \end{cases}$

27. $\begin{cases} a^2x - by = -a \\ xy = 1 + b \end{cases}$

28. $\begin{cases} ax^2 + by^2 = c \\ x^2 - y^2 = d \end{cases}$

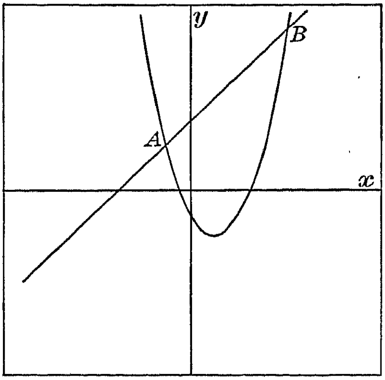
215. Graphic Solutions. The graphic method shown on page 106 can be used for all sets of two equations. The coördinates of the intersection points are the solutions.

216. EXERCISES

The figures on pages 181 to 183 are intended only as suggestions on the general form of the curves; hence no scales are shown on the axes.

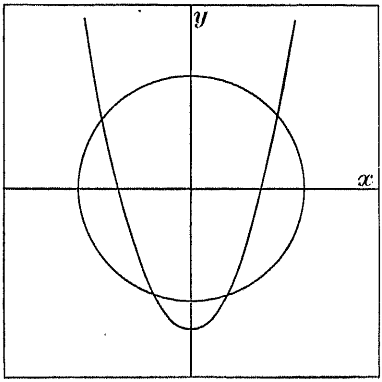
Solve graphically and algebraically the sets:

1. $\begin{cases} y = x^2 - 2x - 1 & (1) \\ y = x + 3 & (2) \end{cases}$



What are the coördinates of A and B in your figure?

On the same axes draw the graph for $y = x - \frac{13}{4}$ and solve this equation with equation (1). How many solutions are there? How is this shown by the graphs?

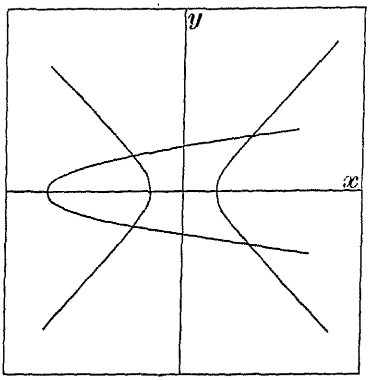


2. $\begin{cases} x^2 + y^2 = 10 & (1) \\ y + 4 = x^2 & (2) \end{cases}$

On the same axes draw the graph of $y + 3 = x^2$. In how many places does this curve cross the curve for equation (1)?

Sketch figures to show how there might be from one to four intersections of a parabola and a circle.

Solve graphically and algebraically:



3. $\begin{cases} x^2 - y^2 = 4 & (1) \\ y^2 - x - 8 = 0 & (2) \end{cases}$

What are the names of these curves? Make some sketches showing the most and the fewest number of intersections that two such curves can have.

(*a*) If the graph of (2) is moved horizontally 6 units to the right, its equation is changed to

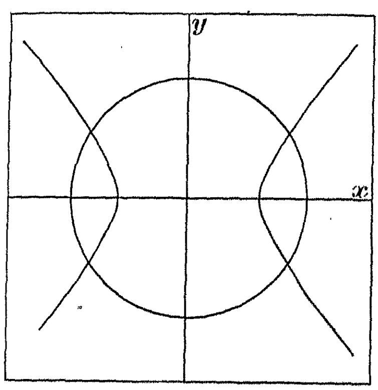
$$y^2 - x - 2 = 0. \qquad (3)$$

Solve equations (1) and (3). How many solutions has this set? How is this shown by the graphs?

(*b*) Consider equation (1) and: $y^2 - x + k = 0.$ (4)

Adding (1) and (4) gives $x^2 - x + k - 4 = 0.$ Find a value of k such that $x = 2$ will be one root of this equation. (See page 167, ex. 22.) Substitute this value of k in (4) and draw the graph. How many intersections are there?

4. $\begin{cases} x^2 + y^2 = 25 & (1) \\ x^2 - y^2 = 9 & (2) \end{cases}$



Where is the center and what is the radius of the circle? The equation

$$(x - 2)^2 + y^2 = 25 \qquad (3)$$

represents a circle for which the center is (2, 0) and the radius is 5.

Without solving the set (2) and (3), tell how many solutions this set has.

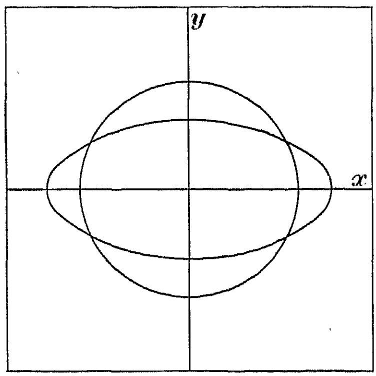
5. $\begin{cases} x^2 + y^2 = 9 & (1) \\ x^2 + 4y^2 = 16 & (2) \end{cases}$

For equation (2) the numerical values of y can be found more easily if the equation is first solved for y; thus, $y = \pm \frac{1}{2}\sqrt{16 - x^2}$. The radicand shows that if x is more than 4 or is less than -4, the value of y is imaginary. For such values of x there is no real point on the curve.

(*a*) Complete the table below, and draw the graph.

$x^2 + 4y^2 = 16$

x	y
-4	0
-3	$\pm \frac{1}{2}\sqrt{7}$, or ± 1.3
-2	$\pm\sqrt{3}$, or ± 1.7
-1	
0	
1	
2	
3	
4	



(*b*) On the same axes draw the graph for

$$x^2 + y^2 = 16 \qquad (3)$$

also for

$$x^2 + y^2 = 4 \qquad (4)$$

In how many points do the graphs of (2) and (3) intersect? The graphs of (2) and (4)?

(*c*) Write the equation of some circle that will not intersect (2) in any point.

The equations studied in this exercise are of the form $ax^2 + by^2 = c$. If a, b, and c are all positive, the curve is an ellipse (or a circle if $a = b$). For other values of the constants a, b, and c, the curve is a hyperbola (or two intersecting straight lines if $c = 0$).

(Supplementary Topic, Pages 184 to 187)

217. A knowledge of the methods used below is not necessary to solve most sets of equations, but the methods are interesting and frequently shorten the work.

EXAMPLE 1. Solve $\begin{cases} x^2 - y^2 = 21 & (1) \\ x + y = 3 & (2) \end{cases}$

Equation (2) is linear. Notice, however, that the left member of (1) is factorable and one of its factors is the left member of (2). Hence dividing (1) by (2) gives $x - y = 7$. Using this equation and (2) we have replaced the given set by the new set:

$$\begin{cases} x - y = 7 \\ x + y = 3 \end{cases}$$

The new set is said to be *equivalent* to the given set because both sets have the same solutions.

EXAMPLE 2. Solve $\begin{cases} x^3 + y^3 = 65 & (1) \\ x + y = 5 & (2) \end{cases}$

As in example 1, factor the left member of (1) and then notice that (1) can be divided by (2). The new set is

$$\begin{cases} x^2 - xy + y^2 = 13 & (3) \\ x + y = 5 & (2) \end{cases}$$

which is equivalent to the given set. Now solve (2) for x or y and substitute in (3).

EXAMPLE 3. Solve $\begin{cases} x^2 + xy + y^2 = 19 & (1) \\ xy = 6 & (2) \end{cases}$

Add (1) and (2), obtaining $x^2 + 2xy + y^2 = 25$. The left member is a trinomial square. Hence $x + y = \pm 5$.

Multiply (2) by -3, and add to (1) to get another trinomial square, $x^2 - 2xy + y^2 = 1$. Hence $x - y = \pm 1$.

The given set is now replaced by the four sets:

$$\begin{cases} x + y = 5 \\ x - y = 1 \end{cases} \quad \begin{cases} x + y = -5 \\ x - y = 1 \end{cases} \quad \begin{cases} x + y = 5 \\ x - y = -1 \end{cases} \quad \begin{cases} x + y = -5 \\ x - y = -1 \end{cases}$$

Solve the four sets and check the solutions in (1) and (2).

218. EXERCISES

1. Solve example 2 on page 184 by solving equation (2) for y and substituting in equation (1).

Solve the following sets of equations:

2. $\begin{cases} 4x^2 - y^2 = 15 \\ 2x - y = 3 \end{cases}$

3. $\begin{cases} 9x^2 - 4y^2 = -35 \\ 3x + 2y = 7 \end{cases}$

4. $\begin{cases} r + 3t = 3 \\ r^2 + 5rt + 6t^2 = 8 \end{cases}$

5. $\begin{cases} xy + y^2 = 20 \\ x^2 + xy = 5 \end{cases}$

6. $\begin{cases} u^2 - 3uv = 4 \\ uv - 3v^2 = 1 \end{cases}$

7. $\begin{cases} r + s = 3 \\ 2rs + 2s^2 = 15 \end{cases}$

8. $\begin{cases} r^3 + s^3 = 9 \\ r + s = 3 \end{cases}$

9. $\begin{cases} x^3 - y^3 = 63 \\ x - y = 3 \end{cases}$

10. $\begin{cases} x^4 + x^2y^2 + y^4 = 21 \\ x^2 + xy + y^2 = 7 \end{cases}$

11. $\begin{cases} x^2 + xy + 4y^2 = 12\frac{7}{9} \\ xy = 2 \end{cases}$

12. $\begin{cases} s^2 - st + t^2 = 13 \\ st - 4 = 0 \end{cases}$

13. $\begin{cases} y^2 + 2yz + 9z^2 = 17 \\ yz = 2 \end{cases}$

14. $\begin{cases} r^2 - rs + 4s^2 = 4 \\ rs = 1 \end{cases}$

15. $\begin{cases} x^2 + y^2 = 10 \\ x^2 - xy + y^2 = 7 \end{cases}$

16. $\begin{cases} x^2 - 4y^2 = .28 \\ x - 2y = .20 \end{cases}$

17. $\begin{cases} x^2 - y^2 = .28 \\ x + y = 1.40 \end{cases}$

18. $\begin{cases} x^3 - y^3 = 26 \\ x^2y - xy^2 = 6 \end{cases}$

19. $\begin{cases} r^3 + s^3 = 35 \\ r^2s + rs^2 = 30 \end{cases}$

20. $\begin{cases} \dfrac{16}{x^2} - \dfrac{9}{y^2} = 3 \\ \dfrac{4}{x} - \dfrac{3}{y} = 1 \end{cases}$

21. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 9 \\ \dfrac{1}{x} + \dfrac{1}{y} = 3 \end{cases}$

22. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13 \\ \dfrac{1}{xy} = 6 \end{cases}$

23. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 5 \\ \dfrac{1}{x} + \dfrac{1}{y} = 3 \end{cases}$

219. Homogeneous Equations. A quantity is *homogeneous* in x and y if the sum of the exponents of x and y is the same in each term. When a homogeneous quantity is set equal to zero we have a homogeneous equation.

Thus, $x^2 + 6y^2 = 3xy$ is a homogeneous equation; it can be written as $x^2 + 6y^2 - 3xy = 0$. But $x^2 + 6y^2 = 8$ is not homogeneous; neither is $x^2 - 3xy = 8x$.

If a set of equations does not contain a linear equation, examine the set to see whether it contains a homogeneous equation and solve the latter for x or for y.

EXAMPLE 1. Solve $\begin{cases} y^2 - 6xy - 27x^2 = 0 & (1) \\ 2x^2 + y - 5 = 0 & (2) \end{cases}$

Solve (1) by factoring. Then $y = -3x$, and $y = 9x$.
Substitute both values of y in (2):

Substitute $y = -3x$ in (2):	Substitute $y = 9x$ in (2):
$2x^2 - 3x - 5 = 0$	$2x^2 + 9x - 5 = 0$
Then $x = \frac{5}{2}$, and $x = -1$.	Then $x = \frac{1}{2}$, and $x = -5$.
Find y by using $y = -3x$.	Find y by using $y = 9x$.
The solutions are:	The solutions are:
$\boxed{\begin{matrix} x = 2\frac{1}{2} \\ y = -7\frac{1}{2} \end{matrix}}$ $\boxed{\begin{matrix} x = -1 \\ y = 3 \end{matrix}}$	$\boxed{\begin{matrix} x = \frac{1}{2} \\ y = 4\frac{1}{2} \end{matrix}}$ $\boxed{\begin{matrix} x = -5 \\ y = -45 \end{matrix}}$

Notice that there are *four* sets of solutions.

EXAMPLE 2. Solve $\begin{cases} 5x^2 + 3y^2 = 12 & (1) \\ xy + y^2 = 3 & (2) \end{cases}$

Neither equation is homogeneous; but a homogeneous equation can be found by multiplying (2) by -4 and adding the result to (1). The new equation is $5x^2 - 4xy - y^2 = 0$.

The solutions of this equation are $x = y$, and $x = -\frac{1}{5}y$. Next, substitute *each* of these values of x in equation (1).

SUGGESTION. If the pupil has the habit of solving the equation always for y, he will find it helpful to rewrite $5x^2 - 4xy - y^2 = 0$ as $y^2 + 4xy - 5x^2 = 0$ before factoring.

220. If a set contains neither a linear nor a homogeneous equation, try one of the following methods.

EXAMPLE 1. Solve $\begin{cases} x^2 - x - 2y = 0 & (1) \\ y^2 + 2x - 5y = 0 & (2) \end{cases}$

Subtracting (2) from (1) gives an equation that can be factored even though it is not homogeneous. The result is $x^2 - y^2 - 3x + 3y = 0$, or $(x - y)(x + y - 3) = 0$.

Hence $x = y$, and $x = 3 - y$.

EXAMPLE 2. Solve $\begin{cases} x^2 + y^2 = 26 & (1) \\ xy + x + y = -1 & (2) \end{cases}$

Multiplying the second equation by 2, and adding the result to the first equation, gives

$$x^2 + 2xy + y^2 + 2x + 2y = 24$$

or
$$(x + y)^2 + 2(x + y) - 24 = 0$$

Hence $x + y = -6$ and $x + y = 4$. Each of these equations should then be solved with equation (2).

221. EXERCISES

Solve the following sets of equations:

1. $\begin{cases} x^2 - 3xy + 2y^2 = 0 \\ x^2 - 5y = 6 \end{cases}$

2. $\begin{cases} y^2 - 12xy + 20x^2 = 0 \\ x^2 - y - 24 = 0 \end{cases}$

3. $\begin{cases} x^2 - 5xy + 4y^2 = 0 \\ x + 12 = y^2 \end{cases}$

4. $\begin{cases} 2y^2 - 5xy + 2x^2 = 0 \\ x^2 + xy = 150 \end{cases}$

5. $\begin{cases} 2x^2 + xy = 16 \\ xy + y^2 = 24 \end{cases}$

6. $\begin{cases} x^2 + xy = 2 \\ y^2 - 2xy = 5 \end{cases}$

7. $\begin{cases} x^2 - 8y - 12 = 0 \\ x^2 - 2y^2 - 4y = 6 \end{cases}$

8. $\begin{cases} x^2 - x - 3y = -3 \\ y^2 + x - 5y = -3 \end{cases}$

9. $\begin{cases} x^2 = x + 5y + 60 \\ y^2 + 3x = y + 60 \end{cases}$

10. $\begin{cases} 6x^2 - y - 1 = 0 \\ 2y^2 + 1 = 8x^2 - y \end{cases}$

11. $\begin{cases} x^2 + y^2 = 82 \\ xy + x + y = -1 \end{cases}$

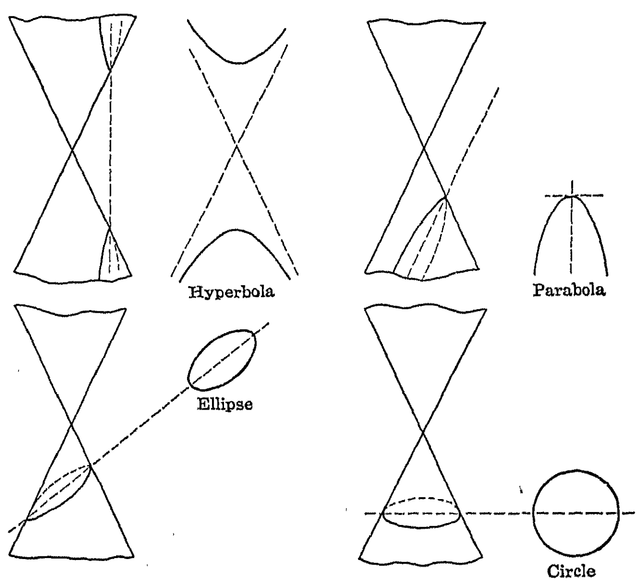
12. $\begin{cases} x^2 + 4y^2 = 37 \\ 2xy - x + 2y = 1 \end{cases}$

13. $\begin{cases} y^2 + 6x^2 = 5xy \\ x^2 - y = 8x - 24 \end{cases}$

14. $\begin{cases} y^2 + 4xy = 5x^2 \\ 2x^2 + y = 3 \end{cases}$

222. Conic Sections. In a field of mathematics called Analytic Geometry, equations are studied and classified according to the nature of their graphs. An equation of the second degree in x and y represents either a parabola, an ellipse, or a hyperbola. The circle is regarded as a special kind of ellipse, and the hyperbola in some cases may be two intersecting lines. All these curves are called *conic sections* because they can be made by cutting a double cone by a plane, as shown in the illustrations below.

Reflectors of searchlights have a parabolic shape. Elliptic and parabolic curves are seen in the designs of many bridges and arches. We know that the path of the earth around the sun is not a circle but an ellipse. Comets move along parabolas or ellipses.



CHAPTER XIV

RADICAL EQUATIONS

223. Equations in which the unknown number occurs under a radical sign are called *radical equations.* When such equations are changed in various ways into linear or quadratic equations, it may happen that, while the algebraic changes are correct, the result does not check in the given equation, as the following example shows.

EXAMPLE

Solve	$\sqrt{x-5}-2=0$		Solve	$\sqrt{x-5}+2=0$	
	$\sqrt{x-5}=2$	(1)		$\sqrt{x-5}=-2$	(1′)
Square (1):	$x-5=4$	(2)		$x-5=4$	(2′)
	$x=9$			$x=9$	
Check.	$\sqrt{9-5}-2=0$		*Check.*	$\sqrt{9-5}+2=0$	
	$\sqrt{4}-2=0$			$\sqrt{4}+2=0$	
	$2-2=0$		But	$2+2\neq 0$	
Hence $x=9$ is a root.			Hence $x=9$ is *not* a root.		

Equations (1) and (1′) differ only in the sign in the right member. This difference disappears when we square; and so equations (2) and (2′) are the same. In (1′), $x = 9$ does not check because $\sqrt{4} = 2$, not ± 2. (See page 122.)

We can see also that the statement $\sqrt{x-5}+2=0$ can never be true because it states that the sum of two positive numbers, $\sqrt{x-5}$ and 2, is zero, which is impossible.

Hence, after solving a radical equation the results must be carefully checked in the given equation. A number that is found as a result of some operations is a root only if it checks in the given equation.

224. The example on page 189 shows the general method for solving radical equations:

Transpose the terms so that one radical expression is the only term in one member of the equation. Square (or cube, or raise to some power) both members of the equation so that the radical in one member is eliminated. Repeat the process until the equation is rational.

If the equation contains more than one radical, transposing and squaring a second time is necessary. This method does not refer to equations like $x^2 + x - \sqrt{3} = 0$ but only to equations in which the *unknown* number is under a radical.

EXAMPLE. Solve $\sqrt{3x-2} - \sqrt{x+3} = 1.$

Transpose $-\sqrt{x+3}$: $\sqrt{3x-2} = 1 + \sqrt{x+3}$

Square both members: $3x - 2 = 1 + 2\sqrt{x+3} + x + 3$

Simplify: $x - 3 = \sqrt{x+3}$

Square both members: $x^2 - 6x + 9 = x + 3$

Simplify: $x^2 - 7x + 6 = 0$

Therefore $x = 6$ and $x = 1$ *may be* roots.

Check 6 and 1 in $\sqrt{3x-2} - \sqrt{x+3} = 1$:

$x = 6,\quad \sqrt{18-2} - \sqrt{6+3} = 1$, or $4 - 3 = 1.$

$x = 1,\quad \sqrt{3-2} - \sqrt{1+3} = 1$, or $1 - 2 \neq 1.$

Hence $x = 6$ is a root, but $x = 1$ is *not* a root.

The values of x that do not satisfy the given equation, although obtained by correct algebraic work, are called *extraneous roots* of the equation.

On page 57, ex. 1, we saw that a root of an equation may be lost by dividing the equation by some factor.

On page 74 we saw that numbers that are not roots of a fractional equation may be introduced if the L. C. M. of the denominators is not properly selected or if the fractions are not in the simplest form.

Here we see that extraneous roots may be introduced when members of an equation are raised to a power.

225. EXERCISES

1. Considering that $\sqrt{a}$ means a *positive* number, which of the following statements are impossible?

(a) $\sqrt{x+1}=2$

(b) $\sqrt{2x-3}=-5$

(c) $\sqrt{x+6}=-2$

(d) $-\sqrt{x+3}-2=\sqrt{x}$

(e) $\sqrt{x}+\sqrt{x+2}=-3$

(f) $-\sqrt{x-1}=2$

Solve the following equations. Check the results.

2. $\sqrt{x+5}=2$
3. $2\sqrt{x-1}=3$
4. $\sqrt[3]{x+4}=-2$
5. $\sqrt[4]{x+3}=2$
6. $3\sqrt{2x-1}=1$
7. $\sqrt{4x+1}=\sqrt{x+10}$
8. $3\sqrt{x-1}=5$
9. $\sqrt[3]{5x+1}=\sqrt[3]{x+13}$
10. $\sqrt{7x+20}=\sqrt{8x+3}$
11. $\sqrt[4]{x+3}=\sqrt{x+1}$

12. Show that the four equations below all lead to the same equation: $x^2-3x-4=0$. In which of the equations is $x=-1$ a root? In which is $x=4$ a root?

(a) $\sqrt{x+5}-\sqrt{3x+4}-1=0$

(b) $\sqrt{x+5}+\sqrt{3x+4}-1=0$

(c) $\sqrt{x+5}-\sqrt{3x+4}+1=0$

(d) $\sqrt{x+5}+\sqrt{3x+4}+1=0$

Solve and check, stating also the extraneous roots:

13. $x+\sqrt{x-5}=11$
14. $y-\sqrt{y-3}=5$
15. $\sqrt{2s-3}=9-s$
16. $2\sqrt{y+1}+y=7$
17. $\sqrt{x+1}+x=5$
18. $2x-\sqrt{x+1}=8$
19. $\sqrt{2x+1}-\sqrt{x}=1$
20. $\sqrt{x}+\sqrt{x+5}=5$
21. $3x+\sqrt{x-1}=7$
22. $\sqrt{4r}-\sqrt{r+9}=3$
23. $\sqrt{2x+5}-\sqrt{x-6}=3$
24. $\sqrt{x+6}-1=\sqrt{3x+7}$
25. $\sqrt{4x-4}-\sqrt{x+8}=3$

Solve and check:

26. $\sqrt{2x+5} = \sqrt{x-1} + 2$

27. $\sqrt{x+7} = \sqrt{x} + 1$

28. $\sqrt{x+1} = 3 + \sqrt{x-8}$

29. $\sqrt{x^2 - x + 2} + x = 4$

30. $8 - \sqrt{x^2 - 5x + 9} = x$

31. $\sqrt{2x+21} = 2 + \sqrt{x+7}$

32. $\sqrt{2x+29} - \sqrt{x+6} = 3$

33. $\sqrt{x-1} = 3 + \sqrt{x-10}$

Solve:

34. $ay = b\sqrt{a^2 - x^2}$ for x. This is one form of the equation of an ellipse.

35. $t = \sqrt{\frac{2s}{g}}$ for s. s is the distance in feet that a body will fall in t seconds; g is a certain constant.

36. $m = \sqrt{\frac{3h}{2}}$ for h. m is the greatest distance in miles from which a light h feet above sea level can be seen.

37. $t = \pi\sqrt{\frac{l}{g}}$ for l. t is the number of seconds required for one swing of a pendulum whose length in centimeters is l.

38. $r = \sqrt{\frac{3V}{\pi h}}$ for V. V is the volume of a cone whose height is h, and whose base is a circle of radius r.

39. $r = \sqrt{\frac{S}{4\pi}}$ for S. S is the area of the surface of a sphere of radius r.

226. EXERCISES — REVIEW OF CHAPTERS XI TO XIV

Solve the following equations and sets of equations:

1. $x^3 - 9x^{\frac{3}{2}} + 8 = 0$

2. $2x^{-6} + 5x^{-3} = 12$

3. $x^{2n} - 5x^n + 6 = 0$

4. $x^{.2} - x^{.1} = 12$

5. $(x^2 - 6x)^2 + x^2 - 6x = 56$

6. $x^4 - 10x^3 + 25x^2 - 10(x^2 - 5x) = 56$

7. $(2x^2 + x)^2 - 18x^2 - 9x + 18 = 0$

8. $(2x^2 + 5x)^2 - 30x^2 - 75x + 36 = 0$

9. $\dfrac{x + a - 2b}{x + a + 2b} - \dfrac{2x - 2a - b}{2x - 2a + b} = 0$

10. $\dfrac{1}{a - b} + \dfrac{a - b}{x} = \dfrac{1}{a + b} + \dfrac{a + b}{x}$

11. $\begin{cases} ax - (a - b)y = b^2 \\ (a + b)x - by = a^2 - b^2 \end{cases}$

12. $\begin{cases} (a - b)x + (a + b)y = a^2 - b^2 \\ 2ax + 2by = a^2 + 2ab - b^2 \end{cases}$

13. $\begin{cases} (2 + x)(3 + y) = 24 \\ x + y = 6 \end{cases}$

14. $\begin{cases} x^2 + y^2 = a^2 \\ xy = \frac{1}{2}a^2 \end{cases}$

15. $\begin{cases} x + 3y + 2z = 2 \\ x = 6(y - z) \\ 3(x - y) = 2z \end{cases}$

16. $\begin{cases} x^2 + xy = 2a^2 \\ x + y = a \end{cases}$

17. $\begin{cases} x + y + z = 2 \\ x + ay = z + 4a \\ a(x + y) + 2z = a^2 \end{cases}$

Ex. 18 to 23 are based on the supplementary work on pages 184 to 187

18. $\begin{cases} x^2 + xy + y^2 = 7 \\ x^2 - xy + y^2 = 19 \end{cases}$

19. $\begin{cases} x^2 + 2xy = 20 \\ y^2 = 3xy - 8 \end{cases}$

20. $\begin{cases} x^2 + y = 3(y - x) \\ x + y^2 = 5(y - x) \end{cases}$

21. $\begin{cases} 2y^2 - x = 24 \\ x^2 + 16y^2 = 10xy \end{cases}$

22. $\begin{cases} y^2 - xy - 6x^2 = 0 \\ x^2 + 2y = 18 - x \end{cases}$

23. $\begin{cases} x^2 + y + 3x = 60 \\ y^2 + 4x^2 = 5xy \end{cases}$

227. REVIEW — RADICALS

1. When we solve $\sqrt{x+6}-\sqrt{3x-5}=5$ in the usual way, we are led to the values $x=3$ and $x=58$.

Substituting $x=3$ gives $\sqrt{9}-\sqrt{4}=5$. Hence 3 is an extraneous root. If the check read $\sqrt{9}+\sqrt{4}=5$, then 3 would be a root; that is, $x=3$ is a root of the equation $\sqrt{x+6}+\sqrt{3x-5}=5$.

Show, in the same way, by substituting $x=58$, that 58 is a root of $\sqrt{x+6}-\sqrt{3x-5}=-5$.

Solve the following equations. If a root is extraneous state, as in ex. 1, the new equation of which it is a root.

2. $\sqrt{x+2}-\sqrt{x-3}=5$

3. $\sqrt{2x-2}+\sqrt{x}=1$

4. $\sqrt{x-1}-\sqrt{x-4}=3$

5. $\sqrt{5x-1}+\sqrt{x+6}=3$

6. $\sqrt{x+1}+\sqrt{x-6}=1$

The equations in ex. 7 to 13 are *linear*. The coefficients are irrational but the unknown number does not appear under a radical sign. The solution of ex. 7, for example, is $x=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ but this should be changed to $5+2\sqrt{6}$.

Solve the following equations for x and check as usual:

7. $x\sqrt{3}-\sqrt{2}=x\sqrt{2}+\sqrt{3}$

8. $x\sqrt{5}+\sqrt{2}(x-3)=\sqrt{5}(x-1)$

9. $x\sqrt{5}+x\sqrt{3}=\sqrt{5}(x+\sqrt{2})$

10. $(x-1)\sqrt{2}+(x+1)\sqrt{3}=x(\sqrt{3}-\sqrt{2})$

11. $\sqrt{3}(x-\sqrt{2})-\sqrt{2}(x-\sqrt{3})=\sqrt{3}-\sqrt{2}$

12. $(x-1)\sqrt{a}+(x+1)\sqrt{b}=\sqrt{a}+3\sqrt{b}$

13. $\sqrt{a}(x-a)+\sqrt{b}(x-b)=a\sqrt{b}+b\sqrt{a}$

228. REVIEW — LITERAL PROBLEMS

1. The perimeter of a rectangle is p. If the length is multiplied by k, the perimeter is increased by a. Find the length and the width of the rectangle.

2. The side of a square is s feet. How many feet must be added to the length and to the width to make it into a square with an area of b square feet?

3. Find a formula for telling in how many days A and B together can do certain work if it takes A alone a days and B alone b days.

4. A wheel r inches in radius is making n revolutions a minute. Write a formula stating how many feet, f, a point on the rim of the wheel moves in a second.

5. If one number is multiplied by a and another number by b, the sum of the products is s. If, however, the first number is multiplied by b and the second by a, the sum of the products is d. Find the numbers.

6. A mixture of a bushels of wheat and b bushels of corn is worth c cents a bushel for chicken feed; but a mixture of b bushels of wheat and a bushels of corn is worth d cents a bushel. Find the cost of a bushel of wheat; of a bushel of corn.

7. One wheel of a carriage makes n more revolutions than another wheel when they go a distance of s feet. The difference of their circumferences is c feet. What equations must be solved to find each circumference?

8. Using r and t as unknown quantities, write the equations for the problem: A freight train goes a miles an hour slower than a passenger train. The freight train needs b hours more than the passenger train to go m miles. Find the rate and the time of the freight train.

CHAPTER XV

ARITHMETIC AND GEOMETRIC PROGRESSIONS

229. Problem. A boy's wages increased \$100 a year for a period of 10 years. If his wages the first year were \$500, how much did he earn during all the 10 years?

One way of solving this problem would be to add the wages for each year; but when a list of numbers is formed according to a rule like the one in this problem, the sum of the numbers can be found more easily by a formula.

230. A series is a succession of terms formed according to some rule, as:

$$2, 4, 6, \ 8, \cdots \text{ and } a, a + d, a + 2\,d, a + 3\,d, \cdots$$
$$2, 4, 8, 16, \cdots \text{ and } x, x^2, x^3, x^4 \cdots.$$

An **Arithmetic Progression** is a series in which the difference of any successive terms is the same.

Thus, 3, 5, 7, 9, $\cdots$ and 10, 7, 4, 1, -2, -5, $\cdots$ are arithmetic progressions. The difference of two successive terms is denoted by the letter d, and represents how much *larger* any term is than its *preceding* one. Hence, in the first progression $d = 2$, and in the second, $d = -3$.

A **Geometric Progression** is a series in which the quotient of any two successive terms is the same.

Thus, 1, 2, 4, 8, $\cdots$ and 125, 25, 5, 1, $\frac{1}{5}$, $\frac{1}{25}$, $\cdots$ are geometric progressions. The quotient or ratio of any two successive terms is denoted by the letter r, and is found by dividing any term by the term *preceding* it. Hence, in the first progression $r = 2$, and in the second, $r = \frac{1}{5}$.

231. EXERCISES

The first term of a progression, whether arithmetic or geometric, is usually denoted by the letter a.

State the first 4 terms of the arithmetic progression:

1. $a = 3,\ d = 2$
2. $a = 5,\ d = 4$
3. $a = 7,\ d = 7$
4. $a = 10,\ d = -3$
5. $a = 5,\ d = -2$
6. $a = \sqrt{3},\ d = \sqrt{5}$
7. $a = x^2,\ d = 2x$
8. $a = 1,\ d = -2x$

State the first 4 terms of the geometric progression:

9. $a = 2,\ r = 3$
10. $a = 5,\ r = -3$
11. $a = 6,\ r = \frac{1}{3}$
12. $a = 10,\ r = \sqrt{2}$
13. $a = 1,\ r = x$
14. $a = P,\ r = 1 + i$
15. $a = x^3,\ r = -x$
16. $a = P,\ r = (1 + i)^2$

Which of the following series are arithmetic progressions? geometric progressions? State the value of d or of r:

17. $-3, 3, 9, 15,$
18. $3, 10, 17, 24,$
19. $\sqrt{2}, 2 + \sqrt{2}, 4 + \sqrt{2},$
20. $1, x, x^2, x^3,$
21. $10, 2, -6, -14,$
22. $b, -b^2, b^3, -b^4,$
23. $x, x^{-1}, x^{-3}, x^{-5},$
24. $8, 4, 2, 1,$

Arrange each of the following sets of numbers so as to form either a geometric or an arithmetic progression:

25. $3, 15, -3, 9,$
26. $\sqrt{2}, \sqrt{8}, 3\sqrt{2},$
27. $\frac{1}{2}, \frac{1}{4}, 1, 2,$
28. $4, -4, 8, 0,$
29. $x^{\frac{1}{3}}, x, x^{\frac{2}{3}}, 1,$
30. $x^{-4}, 1, x^{-2}, x^2,$
31. $\frac{11}{12}, \frac{3}{4}, \frac{5}{6}, \frac{2}{3},$
32. $\sqrt{3}, \sqrt{12}, \sqrt{48},$
33. $3\sqrt{2}, -\sqrt{2}, -3\sqrt{2}, \sqrt{2},$
34. $6, 12, 6\sqrt{2}, 12\sqrt{2},$
35. $a, a + 3d, a + d, a + 2d, a + 4d,$
36. $a, ar^2, ar, ar^3, ar^4,$

ARITHMETIC PROGRESSIONS

232. If a denotes the first term and d the difference of successive terms, the progression can be written as

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ a+4\,d,\ \cdots \text{ etc.}$$

The coefficient of d in the 3d term is 2, and in the 4th term is 3; that is, the coefficient of d in any term is 1 less than the number of the term. Hence, in the so-called "nth term" the coefficient of d is $(n-1)$.

The last, or nth, term is represented by the letter l. Hence

$$l = a + (n-1)d$$

This equation is not only a formula for l, but is a relation between 4 numbers, l, a, n, and d. When any 3 of these are known, this equation is used to find the fourth.

EXAMPLE 1. Find the 15th term of: 4, 7, 10, 13, $\cdots$.

Here $a = 4$, $d = 3$, and we wish to find l when $n = 15$.
Substitution in the formula gives $l = 4 + (15-1)3$, or 46.

EXAMPLE 2. Find the number of terms in the progression:

2, 8, 14, 20, $\cdots$, 56, 62, 68.

Here $a = 2$, $d = 6$, and $l = 68$. We wish to find n.
Hence $68 = 2 + (n-1)6$, or $n = 12$.

233. EXERCISES

1. Find the 10th term of 5, 9, 13, 17, $\cdots$
2. Find the 9th term of $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{7}{8}$, $\cdots$
3. Find the 12th term of $\sqrt{3}$, $\sqrt{12}$, $\sqrt{27}$, $\sqrt{48}$, $\cdots$
4. Find the 100th term of 5, 10, 15, 20, $\cdots$
5. Find the 100th term of 1, 3, 5, 7, $\cdots$
6. Find the kth term of $c-2\,b$, c, $c+2\,b$, $c+4\,b$, $\cdots$

Find the number of terms in each of the following progressions:

7. 3, 8, 13, $\cdots$, 243, 248
8. -12, -3, 6, $\cdots$, 114
9. 1, 3, 5, $\cdots$, 99
10. 17, 23, 29, $\cdots$, 203

234. Arithmetic Means. The numbers 5 and 8, when written between 2 and 11, form the progression 2, 5, 8, 11. Hence 5 and 8 are called arithmetic means between 2 and 11.

Arithmetic means between two numbers are numbers that form an arithmetic progression when inserted between the given numbers.

To insert some means between two numbers, find the value of d and then form the terms $a + d$, $a + 2d$, $a + 3d$, $\cdots$.

EXAMPLE. Insert 3 arithmetic means between 2 and 11.

Here $a = 2$, $l = 11$. Also, $n = 5$ because the progression will contain 5 terms: the first term, the 3 means, and the last term.

Hence $11 = 2 + (5 - 1)d$, or $d = 2\frac{1}{4}$.

The progression is $2,\ 4\frac{1}{4},\ 6\frac{1}{2},\ 8\frac{3}{4},\ 11$.

235. EXERCISES

1. If 5 means are inserted between two numbers, how many terms will there be in the progression?

2. (*a*) Insert 2 arithmetic means between 10 and 34.

(*b*) Insert 3 arithmetic means between 10 and 34.

(*c*) Insert 5 arithmetic means between 10 and 34.

Insert 3 arithmetic means between:

3. -6 and 58

4. 3 and -11

5. $\sqrt{18}$ and $\sqrt{162}$

6. h and k

7. A single mean inserted between two numbers is called *the* arithmetic mean of the numbers. By using two general numbers, as h and k, prove that the arithmetic mean of two numbers is half the sum of the numbers. Is the arithmetic mean the same as the *average* of the numbers?

Ex. 8 to 11 may be used to review work on radicals.

Insert 4 arithmetic means between:

8. $\dfrac{5}{2-\sqrt{3}}$ and $\dfrac{20}{\sqrt{3}-1}$

9. $\dfrac{1}{\sqrt{2}-1}$ and $\dfrac{71}{6\sqrt{2}-1}$

10. $\dfrac{5b}{\sqrt{b}+\sqrt{c}}$ and $\dfrac{5b}{\sqrt{b}-\sqrt{c}}$

11. $\dfrac{5b}{\sqrt{b}+1}$ and $\dfrac{5b}{\sqrt{b}-1}$

236. The Sum of n Terms of an Arithmetic Progression. If the progression 2, 5, 8, 11, 14 is written in the reverse order 14, 11, 8, 5, 2 we see that the sum of any term and the one beneath it is 16, which is the sum of the first and last term. Also, there are as many 16's as there are terms in the progression. Hence *twice* the sum of the terms is $5 \cdot 16$ and the sum of the terms is 40.

This plan is used to find a formula for the sum.

If l is the last term of the progression, then the next to the last term is $l - d$, and the term preceding it is $l - 2d$, the next is $l - 3d$, etc. If, therefore, we write the terms in the usual order and also in the reverse order, we have

$$S = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l$$

and

$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a$$

Adding each term to the one beneath it gives $(a + l)$, and there are n such terms. Hence $2S = n(a + l)$. Dividing by 2, we get $S = \frac{n}{2}(a + l)$, a formula for the sum, S, of n terms.

$$\boxed{\begin{aligned} S &= \frac{n}{2}(a + l) \\ l &= a + (n - 1)d \end{aligned}}$$

This set of two equations contains five quantities: a, d, l, n, and S. When any three are known, the other two can be found by solving this set of equations.

Example. Find the sum of the first 100 odd numbers.

The progression is: $1, 3, 5, 7, \cdots$.
Here $a = 1$, $d = 2$, $n = 100$. Hence we must solve

$$l = 1 + (100 - 1)2, \quad S = \tfrac{100}{2}(1 + l)$$

Solve the equations and finish the problem.

Exercise 1. Solve the problem stated on page 196, § 229.

Exercise 2. Prove that $S = \frac{n}{2}[2a + (n - 1)d]$.

237. EXERCISES — ARITHMETIC PROGRESSIONS

1. Find the sum of 8 terms of 3, 7, 11, 15, $\cdots$

2. Find the sum of 10 terms of -5, -2, 1, 4, $\cdots$

3. Find the sum of 12 terms of 2, 8, 14, 20, $\cdots$

4. Find the sum of 20 terms of 4, 9, 14, 19, $\cdots$

5. Find the sum of 12 terms of 1, 3, 5, 7, $\cdots$

6. Show that the sum of the first n odd numbers is n^2.

7. Find the sum of the first 100 even numbers.

8. Show that the sum of the first n even numbers is $n(n+1)$.

9. Find the sum of the first 100 integers.

10. Show that the sum of the first n integers is $\frac{1}{2}n(n+1)$.

11. Find the sum of the first 100 multiples of 6. (The successive multiples of 6 are 6, 12, 18, etc.)

12. Find the sum of the first n multiples of 5. Why is this sum exactly 5 times the sum found in ex. 10?

13. Find the sum of the *even* numbers *between* 19 and 51.

14. Find the sum of the multiples of 3 *between* 53 and 97.

15. Find the sum of all the numbers between 58 and 344 that are exactly divisible by 7.

16. (*a*) Find the sum of 50 terms of 132, 129, 126, $\cdots$.

(*b*) Find the sum of 39 terms of the same progression.

Why are the sums in (*a*) and (*b*) the same?

17. Write an equation stating that the 3d term of a progression is 13, and another equation stating that the 7th term is 29. Solve the equations for a and d.

What is the 12th term of the progression?

18. The 5th term of a progression is -2, and the 9th term is 18. Find the 11th term. (Use the method in ex. 17.)

19. The 6th term of a progression is $2h+5k$; the 8th term is $3k$. Find the 3d term.

20. The 5th term of a progression is $\sqrt{2}$; the 8th term is $\sqrt{2}+3\sqrt{3}$. Find the sum of the first 10 terms.

On page 200 it was stated that when any three of the numbers a, d, l, n, and S are given, the other two can be found. The ten ways in which the three numbers can be selected are shown below in ex. 21 to 27, 29, 31, 33.

Ex. 21 to 27 lead to easy linear equations. Ex. 27 to 30 lead to sets of linear equations. Ex. 31 to 36 lead to sets requiring the solution of a quadratic equation.

21. Given $a = 3$, $l = 27$, $n = 13$; find d and S.

22. Given $a = 4$, $d = 3$, $l = 37$; find n and S.

23. Given $a = 6$, $l = 42$, $S = 240$; find d and n.

24. Given $d = 2$, $l = 19$, $n = 8$; find a and S.

25. Given $a = 10$, $d = 2$, $n = 15$; find l and S.

26. Given $a = 9$, $n = 12$, $S = 438$; find d and l.

27. Given $d = 4$, $n = 8$, $S = 136$; find a and l.

28. Given $d = -2$, $n = 12$, $S = -36$; find a and l.

29. Given $l = 57$, $n = 12$, $S = 354$; find a and d.

30. Given $l = -28$, $n = 20$, $S = -180$; find a and d.

31. Given $l = 12$, $d = -4$, $S = 208$; find a and n.

32. Given $l = 40$, $d = 5$, $S = 150$; find a and n.

33. Given $a = 15$, $d = -3$, $S = 27$; find n and l.

34. Given $a = 5$, $d = 4$, $S = 90$; find n and l.

35. Given $l = 12$, $d = -4$, $S = 100$; find a and n.

36. Given $a = 21$, $d = -3$, $S = 66$; find n and l.

37. A clerk was promised a salary of \$1200 a year with an increase of \$60 each six months or an increase of \$120 each year. Which was the better offer for the clerk?

Suggestion. Find the total earnings for 10 yr. under each plan. The increase in salary is stated on an annual basis whether the clerk draws the salary for a year or for half a year. When the annual salary is \$1260 the clerk earns \$630 during the half year.

38. If l_{13} and l_6 denote the 13th and the 6th terms of an arithmetic progression, prove that $l_{13} - l_6 = 7\,d$, or, in more general language, $l_n - l_m = (n - m)d$.

238. PROBLEMS

1. An object falls 16 ft. the first second, 48 ft. the second second, 80 ft. the third second, etc. How far will it fall in 12 sec.?

2. How long is a hillside if it takes 13 sec. for a sled to reach the bottom by going 3 ft. the first second, 5 ft. the next second, 7 ft. the next, etc.?

3. A sled, after reaching the bottom of a hill, moves 27 ft. the first second and $1\frac{1}{2}$ ft. less each succeeding second. In how many seconds will the sled stop?

4. If 153 fence posts are to be piled so that the top layer shall contain 1 post, the next layer 2 posts, the next 3 posts, etc., how many posts must be placed in the bottom layer?

5. In a peanut race each player starts from a mark and brings back, one at a time, 10 peanuts. The first peanut is 5 ft. from the mark and each of the others is 5 ft. farther than the preceding. How many feet does each player go in gathering the 10 peanuts?

6. The cost of drilling a well was $1.00 for the first yard, $1.25 for the second, $1.50 for the third, etc. How much did it cost to drill 50 ft.?

7. A ball falls from a height of 30 in. On each rebound it reaches a height 3 in. below the height the time before. Find the total distance up and down through which the ball moved. Ans. 300 in.

8. A Vacation Savings Club collects 2 cents the first week, 4 cents the next week, 6 cents the next, etc. How much is collected in 50 weeks?

9. In a plan like that in ex. 8 (the weekly increase being the same as the first payment), what are the payments if $63.75 is to be collected in 50 weeks?

10. A clock strikes only the hours. How many strokes does it make in a day?

GEOMETRIC PROGRESSIONS

239. By definition (page 196) the successive terms of a geometric progression are made by using the first term, a, and multiplying each term by r to form the next term:

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots,\ \text{etc.}$$

The exponent of r in the third term is 2, in the fourth term is 3, etc.; that is, the exponent of r in any term is 1 less than the number of the term. Hence the nth term is

$$l = ar^{n-1}$$

The symbol t_n is also used for the nth, or last, term of a geometric progression. Subscripts, like $_n$, are very convenient in some problems and are much used in all kinds of mathematical work.

EXAMPLE. Find the 10th term of: 8, 4, 2, 1, $\cdots$

Substitute $a = 8$, $r = \frac{1}{2}$, $n = 10$ in the formula above.

l, or $t_{10} = 8(\frac{1}{2})^9 = 8(\frac{1}{512}) = \frac{1}{64}$

240. EXERCISES

Write the first 6 terms of the progressions:

1. $2, \frac{4}{3}, \frac{8}{9}$
2. $6, -3, \frac{3}{2}$
3. $5\sqrt{2}, 10, 10\sqrt{2}$
4. $P, P(1+i), P(1+i)^2$
5. $\sqrt{3}, \frac{1}{2}\sqrt{6}, \frac{1}{2}\sqrt{3}$
6. $16, -4, 1$
7. Given $a = 4$, $r = 3$, find the 6th term.
8. Given $a = 6$, $r = \frac{1}{2}$, find the 11th term.
9. Given $a = \dfrac{2}{\sqrt{3}-1}$, $r = \sqrt{2}$, find the 9th term.
10. Find the 8th term of 6, $-$ 12, 24.
11. Find the 10th term of $\dfrac{1}{2\sqrt{2}}, \dfrac{1}{2}, \dfrac{\sqrt{2}}{2}$.
12. Find the 7th term of $a, \dfrac{b}{a}, \dfrac{b^2}{a^3}$.
13. Find the 6th term of .4, .04, .004.

241. Geometric Means. The numbers 6 and 18, if written between 2 and 54, form the progression 2, 6, 18, 54. Hence 6 and 18 are called geometric means between 2 and 54.

Geometric means between two numbers are numbers that form a geometric progression when inserted between the given numbers.

To insert geometric means between two numbers, find the value of r, and then write the terms ar, ar^2, ar^3, $\cdots$, etc.

EXAMPLE. Insert 3 geometric means between 2 and 162.

The progression will have 5 terms; hence $n = 5$. Also $a = 2$, and $t_5 = 162$. Then $162 = 2 \cdot r^{5-1}$, or $81 = r^4$; $r = 3$ and -3.

One possible progression is:	2,	6,	18,	54,	162.
Another possibility is:	2,	-6,	18,	-54,	162.

242. EXERCISES

In the work of this chapter use only real values of r.

Insert 2 geometric means between:

1. 2 and 54

2. 5 and 40

3. x^{10} and x^{19}

4. x and x^2

Insert 3 geometric means between:

5. 10 and 160

6. $\sqrt{2}$ and $\sqrt{162}$

7. x^4 and x^6

8. x^{-3} and x^{-1}

9. Prove that the geometric mean of two numbers, as h and k, is the square root of their product.

10. Write two equations, one stating that the 4th term of a geometric progression is 16, and a second equation stating that the 7th term is 128. Solve the equations.

What is the 3d term of the progression? the 6th term?

11. The 5th term of a geometric progression is 8, and the 9th term is 128. Find the 3d term. Find the 12th term.

12. The sum of the 1st and 3d terms of a geometric progression is 10. The 2d term is -4. Find a and r.

13. The 1st term of a geometric progression is a, and the 5th term is b. Find the other terms.

243. The Sum of n Terms of a Geometric Progression. If each term of the equation $S = 1 + 3 + 9 + 27 + 81$ is multiplied by 3, we get $3\,S = 3 + 9 + 27 + 81 + 243$. The numbers 3, 9, 27, and 81 appear in both equations. If the upper equation is subtracted from the lower, the result is $2\,S = 243 - 1$, or $S = 121$.

The same plan is used to find the sum, S_n, of n terms. If we write first, the sum of n terms, and second, the sum multiplied by r, we get the two equations:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \\ rS_n &= ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

If the upper equation is subtracted from the lower, there remains in the right member only the term a from the upper line and the term ar^n from the lower line. Hence

$$rS_n - S_n = ar^n - a, \text{ or } (r-1)S_n = a(r^n - 1), \qquad \text{or}$$

$$S_n = a\frac{r^n - 1}{r - 1} \qquad\qquad S_n = a\frac{1 - r^n}{1 - r}$$

The first form is best when $r > 1$, the second when $r < 1$.

EXAMPLE. Find the sum of 6 terms of 16, 8, 4, 2.

Substitute $n = 6$, $a = 16$, $r = \frac{1}{2}$ in the formula for S_n:

$$S_6 = 16\frac{1 - (\frac{1}{2})^6}{1 - \frac{1}{2}} = 16\frac{1 - \frac{1}{64}}{1 - \frac{1}{2}} = 31\tfrac{1}{2}.$$

244. EXERCISES

Find:

1. S_6 for 2, 4, 8
2. S_7 for 2, $-$ 4, 8
3. S_8 for 16, 8, 4
4. S_6 for $\frac{16}{9}$, $\frac{4}{3}$, 1
5. S_6 for $\sqrt{3}$, 1, $\frac{1}{3}\sqrt{3}$
6. S_7 for $\frac{2}{3}$, 1, $\frac{3}{2}$
7. S_8 for 5, 2.5, 1.25
8. S_7 for 3, 1, $\frac{1}{3}$
9. S_5 for .1, .01, .001
10. S_9 for $\sqrt{2}$, $\sqrt{6}$, $\sqrt{18}$
11. Show that: (a) $S_n = \dfrac{ar^n - a}{r - 1}$; (b) $S_n = \dfrac{rl - a}{r - 1}$.

245. EXERCISES — FACTORING $a^n \pm b^n$

Ex. 1 to 4 should be worked together as a group; likewise ex. 5 to 11.

1. Using the formula for S_n, show that

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

2. In ex. 1, replace x by $\frac{b}{a}\cdot$ Then multiply the left member by a^{n-1}. To multiply the right member by a^{n-1}, multiply the numerator by a^n and the denominator by a. Hence show that

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

3. Using the idea developed in ex. 2, factor:

(*a*) $a^5 - b^5$ (*b*) $x^6 - y^6$ (*c*) $r^7 - s^7$ (*d*) $u^8 - v^8$

4. Which of the quantities in ex. 3 can be factored in some other way?

5. Using the formula for S_n, show that

$$1 - x + x^2 - x^3 + \cdots \quad \pm x^{n-1} = \frac{1 - (-x)^n}{1 - (-x)}$$

6. If n is an *odd* number, does $(-x)^n$ equal $-x^n$ or x^n? If n is an *even* number, does $(-x)^n$ equal $-x^n$ or x^n?

7. Show that if n is an *odd* number,

$$1 + x^n = (1 + x)(1 - x + x^2 - x^3 + \cdots + x^{n-1})$$

8. Show that if n is an *even* number,

$$1 - x^n = (1 + x)(1 - x + x^2 - x^3 + \cdots - x^{n-1})$$

9. In ex. 7 and 8 replace x by $\frac{b}{a}$ and show that

if n is odd, $(a^n + b^n) = (a + b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1})$

if n is even, $(a^n - b^n) = (a + b)(a^{n-1} - a^{n-2}b + \cdots - b^{n-1})$

10. Using the ideas developed in ex. 9, factor:

(*a*) $a^5 + b^5$ (*b*) $x^7 + y^7$ (*c*) $r^8 - s^8$ (*d*) $u^4 - v^4$

11. Which of the quantities in ex. 10 can be factored in some other way?

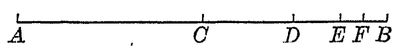
246. The Sum of an Infinite Number of Terms of a Decreasing Progression. If we add more and more of the terms in the progression

$$2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \cdots$$

the sum, S_n, obviously becomes larger and larger. However, no matter how many terms we add, the sum can never become larger than 4. We can see this geometrically from the figure below.

AB represents 4 in., and AC is half of it. CD is half of the remainder; DE is half of the next remainder, etc. Then the sum $AC + CD + DE + EF + \cdots$ is the same as the sum of the terms in the progression above.

From the figure we see that as we add more and more of the parts, the sum gets closer and closer to the number 4



but can never be more than 4; also, by taking enough parts we can make the sum as close to 4 as we wish.

In the above progression $r = \frac{1}{2}$. Only if r is numerically less than 1 is it possible to find a number *beyond which* the sum cannot go and *to which* we can come as close as we please by adding a sufficient number of the terms.

The symbol for the limiting number is S_∞ or just S.

When the number of terms, n, becomes larger and larger we say *n becomes infinite* or *n increases indefinitely*, and the progression itself we call an *infinite progression*. Also we say *S_n approaches a number* rather than "S_n comes closer and closer to a number."

If, as n increases indefinitely, the sum S_n approaches a number, S, and can be made as near to S as we choose, then S is called the *limit* of S_n. This limit is also called the *sum* of the infinite progression.

247. Formula for the Sum. If the terms of a progression become larger and larger, the sum also becomes larger as we add more terms. Hence we shall deal only with *decreasing* progressions; that is, those in which r is numerically less than 1.

The formula (page 206) for the sum of n terms can be written as the difference of two fractions:

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \tag{1}$$

In examining the second numerator, ar^n, remember that any power of a number smaller than 1 is less than the number itself. Thus, $(\frac{1}{2})^2$ is less than $\frac{1}{2}$; and $(\frac{1}{2})^3$ is less than $(\frac{1}{2})^2$. In fact, $(\frac{1}{2})^n$ approaches zero as n increases. Hence, if $r < 1$ then r^n approaches zero as n increases; likewise ar^n also approaches zero.

Therefore, as n increases, the numerator of the second fraction in (1) approaches zero; the denominator does not change. Hence the entire second fraction approaches zero as n increases. The first fraction does not change at all since it does not contain the number n. Hence

As n increases, S_n approaches the value $\frac{a}{1-r}$.

If $r < 1$, then $\quad \mathbf{S = \frac{a}{1-r}}$

EXAMPLE 1. Find the sum of: 9, 3, 1, $\frac{1}{3}$, $\cdots$

Substitute $a = 9$, $r = \frac{1}{3}$ in the formula: $S = \frac{a}{1-r} = \frac{9}{1-\frac{1}{3}} = 13\frac{1}{2}$

EXAMPLE 2. Write as a fraction the decimal .41414141 $\cdots$

Such a decimal is called a *repeating decimal.* It can be written as the sum of the fractions $\frac{41}{100} + \frac{41}{10{,}000} + \frac{41}{1{,}000{,}000} + \cdots$

This is a geometric progression for which $a = .41$ and $r = .01$.

Hence $S = \frac{.41}{1-.01} = \frac{41}{99}$. Therefore $.41414141 \cdots = \frac{41}{99}$.

248. EXERCISES

Find the sum of the following progressions:

1. $4 + 1 + \frac{1}{4} + \cdots$
2. $1 + .1 + .01 + \cdots$
3. $3 - 1 + \frac{1}{3} - \cdots$
4. $5 + 1 + .2 + \cdots$
5. $100 - 10 + 1 - \cdots$
6. $3 + \sqrt{3} + 1 + \cdots$
7. $\sqrt{5} - 1 + \frac{1}{5}\sqrt{5} - \cdots$
8. $6 + 4 + 2\frac{2}{3} + \cdots$

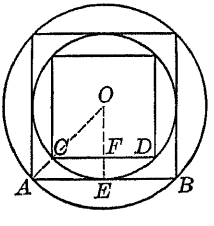
Express as fractions the following repeating decimals:

9. $.15151515 \cdots$
10. $4.15151515 \cdots$
11. $.103103103 \cdots$
12. $2.1636363 \cdots$
13. $.00111111 \cdots$
14. $3.61242424 \cdots$

15. A particle moves in a straight line with such a speed that during any second it moves 75% as far as it did during the preceding second. If it moves 40 ft. in the first second, what is the greatest distance the particle can get from its starting point?

16. A sled after reaching the foot of a toboggan goes 50 ft. the first second. If each second thereafter it goes $\frac{4}{5}$ as far as in the previous second, what is the greatest distance the sled can go?

17. A wheel whose circumference is 5 ft. makes 40 revolutions the first second. If the number of revolutions each second thereafter is 90% of the number for the previous second, how many feet will a point on the rim have moved by the time the wheel stops?



18. In the figure at the left, a square is inscribed in a circle, then a circle is inscribed in that square, etc. Call $AB = a$, and show that the radii of the successive circles form a decreasing geometric progression.

249. Historical Note. The pupil of to-day does not study the various topics in mathematics in the order in which they were developed historically. Thus, he learns to solve quadratic equations and sets of equations before he studies progressions. The Babylonians, however, were acquainted with progressions between 2300 and 1600 years B.C. One tablet records the size of the illuminated portion of the moon for successive days after the new moon. Assuming the entire area of the moon to consist of 240 parts, the Babylonians stated the portion for the first five days as the geometric progression 5, 10, 20, 40, 80, and for the next fifteen days as the arithmetic progression 96, 112, 128, etc.

As we add more and more terms of a progression, the sum of the terms increases numerically but, as stated on page 208, a geometric progression whose ratio is between -1 and $+1$ has a limit. In the study of mathematics we meet many infinite series besides the geometric progressions and in each case the important question is whether or not the sum approaches a limit. If not, the sum cannot be used in computations. A series that does have a limit is called *convergent.* It was not until the nineteenth century that mathematicians began to study series extensively and to find various tests for distinguishing a convergent series from others. The work was done by mathematicians from many countries, in particular by Gauss, a German, Abel, a Norwegian, and Cauchy, a Frenchman. The French astronomer La Place made a great deal of use of series in his work on *Celestial Mechanics,* which is a study of the orbits described by planets and other heavenly bodies moving according to certain laws.

CHAPTER XVI

THE BINOMIAL THEOREM

250. The binomial theorem is a set of rules for writing the terms of $(a + b)^n$ without doing the usual multiplying.

If we multiply in the usual way we find that

$$(a + b)^2 = a^2 + 2\,ab + b^2$$
$$(a + b)^3 = a^3 + 3\,a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$$
$$(a + b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5$$

A study of these expansions shows that for $(a + b)^n$:

1. The first term is a^n. The last term is b^n.

2. The exponents of a decrease by 1 from term to term, while the exponents of b increase by 1. Also, the sum of the exponents of a and b in each term is n.

3. The coefficient of the second term is n. If the coefficient of any term is multiplied by the exponent of a in that term, and divided by 1 more than the exponent of b in that term, the result is the next coefficient.

4. The number of terms is $n + 1$. The signs of the terms are positive. If, however, b is replaced by $- b$, then the terms that contain any odd power of b will be negative.

From these observations we conclude that

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}\,a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\,a^{n-3}b^3 + \cdots + b^n$$

EXAMPLE 1. Write the first 5 terms of $(x + 3)^8$.

The 1st term is a^n or x^8

The 2d term is $na^{n-1}b$ or $8x^7 \cdot 3$

The 3d term is $\frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2$ or $\frac{8 \cdot 7}{1 \cdot 2} x^6 \cdot 3^2$

The 4th term is

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 \quad \text{or} \quad \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} x^5 \cdot 3^3$$

The 5th term is

$$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4}b^4 \quad \text{or} \quad \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} x^4 \cdot 3^4$$

$$\therefore (x + 3)^8 = x^8 + 24x^7 + 252x^6 + 1512x^5 + 5670x^4 + \cdots$$

EXAMPLE 2. Write the first 4 terms of $(x^2 + 2y)^{10}$.

It is a good plan to write first only

$$(x^2)^{10} + \qquad (x^2)^9(2y) + \qquad (x^2)^8(2y)^2 + \qquad (x^2)^7(2y)^3 +$$

Then write the proper coefficients in the blank spaces, and simplify.

251. EXERCISES

Find in the simplest form the first 4 terms of:

1. $(x + 3y)^6$
2. $(x^2 + \frac{1}{2}y)^7$
3. $(x^2 - \frac{1}{3}y)^8$
4. $(x^2 + 3y^2)^8$
5. $(\frac{1}{2}x - y^2)^6$
6. $(3x + 4y^2)^7$
7. $\left(x + \frac{1}{y}\right)^8$
8. $\left(\frac{x}{y} + x\right)^6$
9. $\left(\frac{x}{y} - \frac{y}{x}\right)^7$
10. $\left(\frac{x^2 + 1}{x}\right)^8$ In ex. 10, find the first 4 terms of $(x^2 + 1)^8$ and divide each by x^8; or, write the given quantity as $\left(x + \frac{1}{x}\right)^8$ and expand as in ex. 7 to 9.
11. $\left(\frac{x^2 - 1}{x}\right)^9$ 12. $\left(\frac{y^2 - 1}{y^2}\right)^7$ 13. $\left(\frac{x^3 - x}{y}\right)^8$

252. The rth Term of $(a + b)^n$. To write any required term in an expansion without writing all the terms that precede it, we need a formula for the rth term.

According to the rules on page 212, we know that

the 4th term is $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3$

the 5th term is $\dfrac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4}b^4$

the 6th term is $\dfrac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^{n-5}b^5$

A study of these terms shows that in the rth term:

1. The exponent of b is 1 less than the number of the term, or $r - 1$. Since the sum of the exponents in any term is n, the exponent of a must be $n - (r - 1)$, or $n - r + 1$.

2. The numerator of the coefficient is the product of the $(r - 1)$ successive integers decreasing from n.

3. The denominator of the coefficient is the product of the integers up to and including $r - 1$. If we count the number 1, then the denominator also consists of $(r - 1)$ successive integers.

Hence the rth term (r not equal to 1) is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} a^{n-r+1} b^{r-1}$$

If b is negative and $r - 1$ is an odd number, the sign of the term is negative. In all other cases it is positive.

We will find it easier to remember the rules if we fix our attention on the number $(r - 1)$. Notice that in the coefficient there are $(r - 1)$ factors in the numerator, and $(r - 1)$ factors in the denominator. The exponent of b is $(r - 1)$, and of a is $n - (r - 1)$.

EXAMPLE. Write the 6th term of $(4x + 7y)^8$.

Since $r - 1 = 5$, the 6th term is $\dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(4x)^3(7y)^5 = ?$

253. EXERCISES

Write:

1. 5th term of $(x + 1)^8$
2. 5th term of $(x^2 + 3)^7$
3. 4th term of $(x + 3y^2)^8$
4. 7th term of $(x + b)^{12}$
5. 6th term of $(x + 2b)^{12}$
6. 6th term of $(2x + y)^{15}$
7. 5th term of $(x^2 - y)^{12}$
8. 4th term of $(x - y^2)^{10}$
9. 7th term of $(\frac{1}{2}x + y)^{10}$
10. 6th term of $(a + \frac{1}{2}b)^9$

11. Write the term of $(x + 2)^{11}$ that contains x^7.

HINT. The following argument shows that the 5th term contains x^7. In the formula on page 214 for the rth term, the exponent of x is $(n - r + 1)$. We therefore wish $n - r + 1$ to equal 7. Since $n = 11$, we see that $11 - r + 1 = 7$, or $r = 5$.

12. Write the term of $(x^2 + 1)^{10}$ that contains x^8.

HINT. Substitute y for x^2 and, as in ex. 11, find the term that contains y^4. Or, as in ex. 2, page 213, note that the powers of x are:

$$(x^2)^{10} + \cdots (x^2)^9 + \cdots (x^2)^8 + \cdots (x^2)^7 + \cdots.$$

Which term will contain x^8? Find that term.

13. Write the term of $(x^3 + 1)^8$ that contains x^{15}.

14. Write the term of $(x^4 - 1)^9$ that contains x^{20}.

15. Write the term of $(x^2 + 3)^8$ that contains x^{10}.

254. The Factorial Notation. The symbol $3!$ means the product of the integers 1, 2, 3. It is read "factorial 3." Similarly, $4!$ means $1 \cdot 2 \cdot 3 \cdot 4$; and $5!$ means $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$.

Likewise, $n!$ means the product of the successive integers from 1 to n inclusive; that is,

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-2)(n-1)n$$

The factorial notation is convenient in writing various formulas. Thus, the rth term of the binomial expansion is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}\,a^{n-r+1}b^{r-1}$$

However, when the actual value of the coefficient is wanted, we replace $(r-1)!$ by $1 \cdot 2 \cdot 3 \cdot \cdots \cdot (r-1)$ so that we can use cancellation in simplifying the fraction.

255. Compound Interest. If i is the rate of interest on P dollars, the interest for one year is Pi, and the amount for one year is $P + Pi$, or $P(1 + i)$.

To find the amount due at the end of any year, multiply the principal for that year by $(1 + i)$.

In previous work the letter r, for various reasons, was used to denote the rate of interest. Hereafter the letter i will be used in order to agree with the most modern practice of actuaries.

Hence the amount due at the end of the second year is $(1 + i)$ times $P(1 + i)$, or $P(1 + i)^2$; and the amount at the end of the third year is $P(1 + i)^3$; etc.

The amount, A, due on P dollars after n years with interest at i% compounded annually is: $A = P(1 + i)^n$.

Thus, the amount due on \$5000 after 10 yr. at 6% compounded annually is $5000(1 + .06)^{10}$, or $5000(1.06)^{10}$.

The binomial theorem can be used to find the value of $(1.06)^{10}$. For example:

$$(1 + .06)^{10} = 1^{10} + 10(1)^9(.06) + 45(1)^8(.06)^2 + 120(1)^7(.06)^3 + 210(1)^6(.06)^4 + 252(1)^5(.06)^5 + \cdots$$

To find the value to the nearest thousandth we must add terms until the third decimal place is no longer influenced.

The value is 1.791 because 1.7908 is nearer to 1.791 than to 1.790.

$$\begin{aligned} .06^2 &= .0036 \\ .06^3 &= .000216 \\ .06^4 &= .00001296 \\ .06^5 &= .0000007776 \end{aligned}$$

$$\begin{aligned} 1^{10} &= 1. \\ 10(.06) &= .6000 \\ 45(.06)^2 &= .1620 \\ 120(.06)^3 &= .02592 \\ 210(.06)^4 &= .00272\cdots \\ 252(.06)^5 &= .00019\cdots \\ & \overline{1.7908} \end{aligned}$$

256. EXERCISES

Find to the nearest thousandth the value of:

1. 1.06^8
2. 1.06^4
3. 1.06^7
4. 1.05^7
5. 1.05^8
6. 1.05^{10}
7. 1.04^{10}
8. 1.03^{10}
9. 1.08^6

(Supplementary Topics, Pages 217 to 219)

257. Proof of the Binomial Theorem. The proof of this theorem for positive integral exponents is based on a method called *mathematical induction*, and proceeds thus:

1. If we square $(a + b)$ by the usual method, we find that

$$(a + b)^2 = a^2 + 2\,ab + b^2$$

This expression can also be obtained by putting $n = 2$ in the formula on page 212.

Hence the theorem is true at least for the value $n = 2$.

2. We next prove that if the theorem is true for any one power, then it is true also for the next higher power. In other words, if we substitute $n + 1$ for n in the formula on page 212, we should get the same result as if we multiplied the expression for $(a + b)^n$ by $(a + b)$. We can prove this by actually doing the multiplying. Multiplying the expression for $(a + b)^n$ by a, we get the terms shown in line (1) below; and multiplying by b, we get the terms shown in line (2):

$$\begin{array}{ll} (1) & a^{n+1} + na^nb + \dfrac{n(n-1)}{1 \cdot 2}a^{n-1}b^2 + \cdots \\ (2) & \qquad\quad a^nb + \qquad\quad na^{n-1}b^2 + \dfrac{n(n-1)}{1 \cdot 2}a^{n-2}b^3 + \cdots \end{array}$$

If these two quantities are added by combining similar terms, we find that the sum is exactly the same as if we had written $n + 1$ for n in the formula on page 212.

Part 1, above, shows that the theorem is true if $n = 2$. Part 2 shows that if the theorem is true for $n = 2$, then it is true also for $n = 3$. By repeating this argument we see that the theorem is true for any positive integer.

A proof by mathematical induction involves always two steps. First, prove that the theorem is true for some one integer. Second, prove that if the theorem is true for any integer, it is true also for the next higher integer.

258. The Binomial Theorem for Fractional Exponents. By methods that cannot be explained here, it is proved that when $a > b$ the binomial theorem may also be used for fractional and even negative exponents. In such cases there is not a definite number of terms in the expansion, for none of the coefficients becomes zero. Nevertheless, such expansions are very useful as a means of computing.

Thus, the first 4 terms of $(a+b)^{\frac{1}{2}}$ are

$$a^{\frac{1}{2}} + \tfrac{1}{2}\,a^{\frac{1}{2}-1}b + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}\,a^{\frac{1}{2}-2}b^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}\,a^{\frac{1}{2}-3}b^3.$$

$$\therefore\ (a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}\,a^{-\frac{1}{2}}b - \tfrac{1}{8}\,a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}\,a^{-\frac{5}{2}}b^3 \cdots$$

We can use this as a formula to find approximately $\sqrt{18}$. Substitute $a = 16$, $b = 2$. Then

$$\begin{aligned}(16+2)^{\frac{1}{2}} &= 16^{\frac{1}{2}} + \tfrac{1}{2}\cdot 16^{-\frac{1}{2}}\cdot 2 - \tfrac{1}{8}\cdot 16^{-\frac{3}{2}}\cdot 2^2 + \tfrac{1}{16}\cdot 16^{-\frac{5}{2}}\cdot 2^3\\ &= 4 + \tfrac{1}{2}\cdot\tfrac{1}{4}\cdot 2 - \tfrac{1}{8}\cdot\tfrac{1}{64}\cdot 4 + \tfrac{1}{16}\cdot\tfrac{1}{1024}\cdot 8\\ &= 4 + .250000 - .007812 + .000488\\ &= 4.242676\cdots\end{aligned}$$

In the above expansion for $(a+b)^{\frac{1}{2}}$ we notice that after the first term, the signs of the terms are alternately plus and minus. Such series are called *alternating* series. They are particularly good for purposes of computation because the sum of all the terms after the rth term cannot be more than the rth term itself. Hence, when using such a series, we know at every stage of the work just how big the possible error is. Also, the smaller that b is in comparison to a, the better will the approximation be.

Other approximate formulas are:

1. $(a-b)^{\frac{1}{2}} = a^{\frac{1}{2}} - \tfrac{1}{2}\,a^{-\frac{1}{2}}b - \tfrac{1}{8}\,a^{-\frac{3}{2}}b^2 - \tfrac{1}{16}\,a^{-\frac{5}{2}}b^3$
2. $(a+b)^{\frac{1}{3}} = a^{\frac{1}{3}} + \tfrac{1}{3}\,a^{-\frac{2}{3}}b - \tfrac{1}{9}\,a^{-\frac{5}{3}}b^2 + \tfrac{5}{81}\,a^{-\frac{8}{3}}b^3$
3. $(a+b)^{\frac{2}{3}} = a^{\frac{2}{3}} + \tfrac{2}{3}\,a^{-\frac{1}{3}}b - \tfrac{1}{9}\,a^{-\frac{4}{3}}b^2 + \tfrac{4}{81}\,a^{-\frac{7}{3}}b^3$
4. $(a+b)^{\frac{3}{2}} = a^{\frac{3}{2}} + \tfrac{3}{2}\,a^{\frac{1}{2}}b + \tfrac{3}{8}\,a^{-\frac{1}{2}}b^2 - \tfrac{1}{16}\,a^{-\frac{3}{2}}b^3$

259. EXERCISES

1. By means of the binomial theorem derive the formulas on page 218.

2. In using formula 1 the number a should be so chosen that the power $a^{\frac{1}{2}}$ will be an integer. Likewise in 2, the power $a^{\frac{1}{3}}$ should be an integer. How should the number a be chosen when using formula 3? formula 4?

By the approximation formulas, find to six decimals:

3. $\sqrt{17}$ 4. $\sqrt[3]{28}$ 5. $\sqrt{37}$

6. $\sqrt{35}$ Here it is better to say $35 = 36 - 1$ rather than $35 = 25 + 10$, so that the second number, b, will be as small as possible compared to a.

7. $(79)^{\frac{1}{2}}$ 10. $(65)^{\frac{2}{3}}$ 13. $(65)^{\frac{3}{2}}$

8. $(66)^{\frac{1}{3}}$ 11. $(62)^{\frac{2}{3}}$ 14. $(101)^{\frac{3}{2}}$

9. $(29)^{\frac{1}{3}}$ 12. $(78)^{\frac{1}{2}}$ 15. $(63)^{\frac{1}{3}}$

260. Historical Note. According to popular belief, Newton (1642–1727) was led to the discovery of the law of gravitation by the falling of an apple. We know as a fact, however, that Newton's attempts to explain some difficulties that Wallis (page 142) had in interpolation led to the discovery of the binomial theorem. He first stated it as a theorem for extracting roots but gave no proof of it. A satisfactory proof was not given until the nineteenth century, and this by Abel (page 211).

Although Newton stated the theorem in its general form, the Arabs and Hindus a thousand years earlier had used simpler cases as a means of extracting square roots and cube roots, and a German mathematician, Stifel (1487–1567), had worked out the coefficients for all powers up to the seventeenth.

The binomial coefficients can be written in a curious way, as shown below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

This arithmetic triangle, sometimes called "Pascal's triangle," was first published by a Frenchman, Pascal, in 1653. Each row of numbers shows the coefficients of a power of $(a + b)$ beginning with the zero power. Thus, the fifth row shows the coefficients of $(a + b)^4$. The numbers in each row can be found from the two numbers, one to the left of it and one to the right of it, in the preceding row. In the fifth row, for example, $4 = 1 + 3$, $6 = 3 + 3$, etc. The numbers in the next row would be

$$\begin{array}{lcccccc} & 1, & (1+4), & (4+6), & (6+4), & (4+1), & 1 \\ \text{or} & 1, & 5, & 10, & 10, & 5, & 1. \end{array}$$

When they are written in this form it is easy to see why the coefficients should repeat themselves in reverse order after the middle term is reached.

CHAPTER XVII

LOGARITHMS

261. Logarithms are Exponents. The use of logarithms in numerical work enables us to perform in a few minutes computations that would otherwise require many hours. The fundamental idea consists in writing numbers as powers of some other number, just as 100 can be written as 10^2.

The laws of exponents stated in Chapter IX are true also for such exponents as are used in this chapter.

We know that $64 = 8^2$ and call 2 the logarithm of 64; but we might also regard 3 as the logarithm of 64 because $64 = 4^3$. Hence we must state what number, as 8 or 4 in this illustration, is being raised to some power. This number we call the *base*. Thus, we say that

The logarithm of 64 to the base 8 is 2 because $64 = 8^2$
The logarithm of 64 to the base 4 is 3 because $64 = 4^3$
The logarithm of 100 to the base 10 is 2 because $100 = 10^2$

We write these statments more briefly as

$$\log_8 64 = 2, \qquad \log_4 64 = 3, \qquad \log_{10} 100 = 2;$$

but when the base is 10, we usually omit any mention of the base and write merely $\log 100 = 2$.

Exercise. Find by trial the values of: $\log 1000$, $\log_3 81$, $\log_2 32$, $\log_5 625$, $\log_5 25$, $\log_{25} 5$.

The equation $N = b^l$ expresses the relation between a number, N, and its logarithm, l, to the base b.

The *logarithm* of a number is the exponent of the power to which another number, called the base, must be raised to equal the given number.

262. ORAL EXERCISES

1. Supply the missing numbers in the sentences:

(*a*) $9 = 3^2$. $\therefore$ The logarithm of to the base is .

(*b*) $8 = 2^3$. $\therefore$ The logarithm of to the base is .

Form similar sentences from the statements:

(*c*) $125 = 5^3$ $\quad .01 = 10^{-2}$ $\quad 4 = 16^{.25}$

(*d*) $2 = 10^{.3}$ $\quad 4 = 10^{.6}$ $\quad 8 = 10^{.9}$

(*e*) $6.5 = 10^{.8129}$ $\quad 65 = 10^{1.8129}$ $\quad 650 = 10^{2.8129}$

2. (*a*) What does 2^0 equal? What does $\log_2 1$ equal?

(*b*) Find $\log_3 1$, $\log_4 1$, $\log_{10} 1$, $\log_b 1$

(*c*) What is the logarithm of 1 to any base?

3. (*a*) Find $\log_2 2$, $\log_3 3$, $\log_4 4$, $\log_b b$

(*b*) In any system of logarithms what is the logarithm of the base itself?

4. (*a*) If the number 10 is raised to some positive power, is the result positive or negative?

(*b*) If the number 10 is raised to some negative power, as 10^{-2} for example, is the result positive or negative?

(*c*) Remembering that $10^{\frac{1}{2}}$ means the positive square root of 10, can we get a negative result by raising 10 to a fractional power?

The answers to (*a*), (*b*), and (*c*) show that there is no logarithm for a negative number because it is impossible to find a power of 10 that will give a negative result.

5. From the equation $65 = 10 \times 6.5 = 10^1 \times 10^{.8129}$ explain why the logarithm of 65 should be just 1 more than the logarithm of 6.5. From a similar equation explain why the logarithm of 650 is 1 more than the logarithm of 65.

6. If 10 is raised to some power between 1 and 2, for example $10^{1.5}$, the result is a number between 10 and 100. If 10 is raised to some power between 2 and 3, the result must lie between what two numbers?

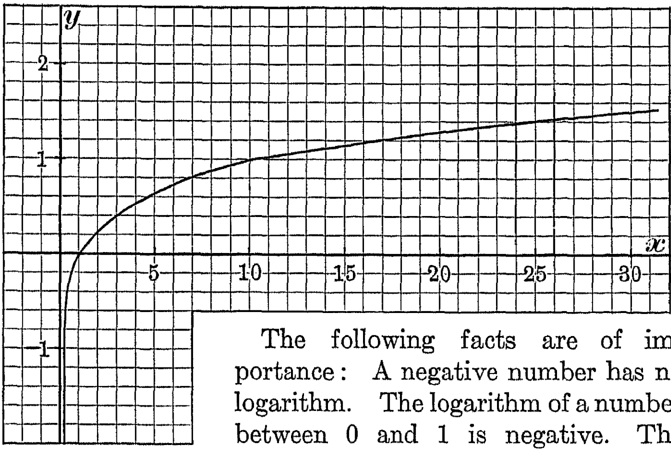
263. Graph of $y = \log x$. To draw the graph we must first find the values of y for some values of x.

$$\begin{array}{llll}
\text{Since } 10^3 & = 1000, & \log 1000 & = 3 \\
\text{Since } 10^2 & = 100, & \log 100 & = 2 \\
\text{Since } 10^1 & = 10, & \log 10 & = 1 \\
\text{Since } 10^0 & = 1, & \log 1 & = 0 \\
\text{Since } 10^{-1} & = .1, & \log .1 & = -1 \\
\text{Since } 10^{-2} & = .01, & \log .01 & = -2 \\
\text{Since } 10^{-3} & = .001, & \log .001 & = -3
\end{array}$$

x	y, or $\log x$
1000	3
100	2
10	1
1	0
.1	-1
.01	-2
.001	-3

We can find some other points on the curve as follows:

$$\begin{array}{llll}
\text{Since } 10^{.5} & = \sqrt{10} = 3.162, & \log 3.162 & = .5 \\
\text{Since } 10^{1.5} & = 10^1 10^{.5} = 31.62, & \log 31.62 & = 1.5 \\
\text{Since } 10^{2.5} & = 10^2 10^{.5} = 316.2 & \log 316.2 & = 2.5
\end{array}$$



The following facts are of importance: A negative number has no logarithm. The logarithm of a number between 0 and 1 is negative. The logarithm of 1 is 0.

EXERCISE. Make an enlarged copy of the above graph. From your graph find the approximate values of log 5, log 15, log 20, log 25.

264. The Characteristic and the Mantissa. The logarithm of 5678 is 3.7542. Without considering how this is known, let us find the logarithms of some other numbers from this one.

$$567.8 = \frac{5678}{10} = \frac{10^{3.7542}}{10} = 10^{2.7542} \quad \log 567.8 = 2.7542$$

$$56.78 = \frac{567.8}{10} = \frac{10^{2.7542}}{10} = 10^{1.7542} \quad \log 56.78 = 1.7542$$

$$5.678 = \frac{56.78}{10} = \frac{10^{1.7542}}{10} = 10^{.7542} \quad \log 5.678 = \ .7542$$

$$.5678 = \frac{5.678}{10} = \frac{10^{.7542}}{10} = 10^{.7542-1} \quad \log .5678 = \ .7542 - 1$$

If we add .7542 and -1 we get $-.2458$, which agrees with the statement on page 223 that a number between 0 and 1 has a negative logarithm. We shall see later that the form $.7542 - 1$ is more useful than $-.2458$. Also

$$.05678 = \frac{.5678}{10} = \frac{10^{.7542-1}}{10} = 10^{.7542-2}; \log .05678 = .7542 - 2.$$

Notice how each division by 10 moves the decimal point in the number and decreases the logarithm by 1.

We see that the logarithm of a number may be thought of as composed of two parts, an integer and a decimal. The integer may be positive, as in 2.7542, or negative, as in $.7542 - 1$. The decimal part should always be positive.

The integral part of the logarithm is called the *characteristic* and the decimal is called the *mantissa.*

Exercise 1. Given that $\log 426.3 = 2.6297$, find by the above method log 42.63, log 4263, log .4263, log .004263, log 42,630, and state the characteristic and the mantissa of each logarithm.

Exercise 2. Given that $\log 3.456 = .5386$, find by the above method log .003456, log 34.56, log 3456, log .3456, log 345,600, log 3,456,000.

265. Rules for the Characteristic. From our work we see that the characteristic of a logarithm need not be printed in a table because we can find it by the following rules:

If a number is larger than 1, *the characteristic is positive and one less than the number of figures to the left of the decimal point.*

If a number is smaller than 1, *the characteristic is negative and numerically one more than the number of zeros between the decimal point and the first non-zero figure.*

EXERCISE. State just the characteristic of the logarithms of 2367, .2367, 23.67, .0002367, .02367, 236,700.

266. Table of Mantissa. On page 224 we learned that

Numbers that differ only in the position of the decimal point have the same mantissa in their logarithms.

For this reason a table need not print the mantissas of 876 and 87.6 and 8.76, etc. It is sufficient to have the mantissa for only one of these numbers.

The table on pages 280 and 281 gives the mantissas of the integers from 100 to 999. For convenience in printing, the decimal point is omitted before each mantissa.

In the upper left-hand corner of the table we find the mantissa for log 100. This mantissa consists of 4 zeros, as it should, since log 100 = 2.0000. The next mantissa to the right is the mantissa for log 101, then for log 102, etc., the last one being for log 109. Beginning on the next line we find the mantissa for log 110, then for log 111, etc.

The column marked D will be explained later.

EXAMPLE 1. Find log 436.

The characteristic is 2. The mantissa is .6395

$$\log 436 = 2.6395$$

EXAMPLE 2. Find log .00719.

The characteristic is -3. The mantissa is .8567

$$\log .00719 = .8567 - 3$$

267. Interpolation of Logarithms. Interpolation is a method for finding the logarithm of a number not given directly in the table, as log 372.5 for example.

Since the number 372.5 is halfway between 372 and 373, we assume (although it is not strictly true) that log 372.5 is halfway between log 372 and log 373.

log 372 = 2.5705 and log 373 = 2.5717. The difference between the two logarithms is .0012 but we need to remember only the figure 12. This number can be found in the column marked D, and is called the *tabular difference.* Half of this difference is 6 (actually it is .0006), and this is added to log 372.

Hence log 372.5 = 2.5705 + .0006 = 2.5711.

EXAMPLE 1. Find log 267.3.

On page 280:
log 267 = 2.4265
$D = 16$

The tabular difference between log 267 and log 268 is 16.

.3 of 16 is 5 (to the nearest whole number).

log 267.3 = 2.4265 + .0005 = 2.4270.

EXAMPLE 2. Find log .006957.

The characteristic is − 3. The mantissa is the same as for log 695.7. Hence we find the mantissa for 695.7.

On page 281:
The mantissa for log 695 is .8420
$D = 6$

The tabular difference is 6.
.7 of 6 = 4 (to the nearest integer).
8420 + 4 = 8424
log .006957 = .8424 − 3

When finding the mantissa for log .6957 or log 69.57 proceed as if you were finding the mantissa for log 695.7; that is, suppose that the decimal point in the number has been shifted so that the number is between 100 and 999. The interpolation will then mean that the tabular difference, D, is multiplied by a certain number of tenths.

268. EXERCISES

Find the logarithms of the following numbers:

1. 415.7 The difference between log 415 and log 416 is 11, but $D = 10$. In such cases use the given value of D as it represents the average difference at that point.

2. 543.6	6. 4.375	10. 10.01
3. 693.2	7. .3263	11. 2.003
4. 56.38	8. .1827	12. 749.8
5. 25.34	9. .08426	13. .9999

NOTE. Later (page 230) it will be seen that the logarithm of .006957 (see ex. 2, page 226) may need to be written in the form $1.8424 - 4$ or $2.8424 - 5$ or $7.8424 - 10$, etc. Many computers prefer the last form. The part -10 is then usually omitted, the work appearing as log .006957 = 7.8424.

269. Antilogarithms. Compare the two statements:

The logarithm of 100 is 2. The antilogarithm of 2 is 100.

In other words, the *antilogarithm* of x is the number of which x is the logarithm.

EXAMPLE. Find the antilog 1.4683; that is, find the number of which 1.4683 is the logarithm.

Disregard the characteristic, 1, and find in the table the mantissa .4683. It is found in the row marked 29 at the left, and in the column marked 4 at the top. Hence .4683 is the mantissa of the logarithms of numbers like

294, 29.4, 2.94, .294, .0294, .00294, etc.

The characteristic determines the correct place for the decimal point. In this case the required number is 29.4.

Antilog 1.4683 = 29.4 because log 29.4 = 1.4683.

When the characteristic is negative it may be necessary to insert zeros to the right of the decimal point. Thus, antilog $.4683 - 1 = .294$; antilog $.4683 - 2 = .0294$.

270. Interpolation of Antilogarithms. If the mantissa of a given logarithm, as 2.7375, is not in the table, the antilogarithm is found by interpolation as follows:

Disregard the characteristic and look in the table for the mantissa nearest to .7375 and smaller than it. Then

	.7372	corresponds	to	546
	.7375	corresponds	to	?
The next mantissa	.7380	corresponds	to	547.

Since .7375 is $\frac{3}{8}$ of the way from .7372 to .7380 we assume that .7375 corresponds to $546\frac{3}{8}$. It is inconvenient, however, to leave the number in this form as we may need to have 4 or more figures to the left of the decimal point. We therefore change $\frac{3}{8}$ to .4 (never using more than the nearest tenth). Hence .7375 is the mantissa of the logarithm of numbers like 546.4, 54.64, 5.464, .5464, etc.

The characteristic determines the correct place for the decimal point. In this case the required number is 546.4.

When the characteristic is negative it may be necessary to insert zeros to the right of the decimal point. Thus, antilog $.7375 - 3 = .005464$; antilog $.7375 - 2 = .05464$.

Suggestion. In finding antilogarithms by interpolation it is always necessary to change a fraction to tenths. This can be done easily by annexing a zero to the numerator and then dividing. To change $\frac{5}{8}$, for example, divide 50 by 8. Only the nearest integer should be used for the quotient.

271. EXERCISES

Find the antilogarithms of:

1. 1.8084	**6.** 3.6593	**11.** .1495
2. $.8084 - 1$	**7.** $.7405 - 1$	**12.** $.8689 - 1$
3. .4719	**8.** $.9166 - 2$	**13.** 1.0182
4. 1.5615	**9.** $.3804 - 3$	**14.** .9998
5. 2.6000	**10.** .2930	**15.** .0004

272. Multiplication by Logarithms.

THEOREM. *The logarithm of the product of two (or more) numbers is the sum of the logarithms of the numbers.*

That is, $\log xy = \log x + \log y$

Proof. Let $x = 10^m$, that is, $\log x = m$

Let $y = 10^n$, that is, $\log y = n$

Then $xy = 10^{m+n}$. $\therefore \log xy = m + n = \log x + \log y$

EXAMPLE. Find the product $690.7 \times .3882$ by logarithms.

$$\begin{aligned} \log 690.7 &= 2.8392 \\ \log .3882 &= \underline{.5890 - 1} \\ \log [\quad] &= 3.4282 - 1 \\ 690.7 \times .3882 &= 268.1 \end{aligned}$$

When added, the characteristic is $3 - 1$, or 2. The antilogarithm of 2.4282 is 268.1, and may be written in a square bracket in the third line of the work.

The antilogarithm of 2.4282 is $268\frac{1}{16}$, or 268.0625, which, to the nearest tenth, is 268.1. The usual multiplication shows that $690.7 \times .3882 = 268.12974$; but the number 268.1 is as close as we can get by four-place logarithms.

When working with a four-place table of mantissas, only four figures should be retained in the result (not counting any zeros that are used solely to fix the decimal point).

The figures thus retained are called *significant figures.*

273. EXERCISES

Find by logarithms the indicated products:

1. 85.7×6.46
2. 24.6×25.3
3. 7.26×3.59
4. 39.2×6.32
5. 89.2×78.4
6. 700.2×6.303
7. 356.4×38.92
8. 40.69×207.1
9. 572.1×84.62
10. 912.8×9.752
11. $.5432 \times .06783$
12. $.3142 \times .2492$
13. $.4139 \times .3412$
14. $.7314 \times .02063$
15. $.1191 \times .02291$

Without using a table of logarithms, prove that

16. $\log 25 + \log 4 = 2$
17. $\log \frac{5}{16} + \log \frac{8}{3} + \log \frac{6}{5} = 0$

274. Division by Logarithms.

THEOREM. *The logarithm of a fraction is the logarithm of the numerator minus the logarithm of the denominator.*

That is, $\log \frac{x}{y} = \log x - \log y$

Proof. Let $x = 10^m$, that is, $\log x = m$
Let $y = 10^n$, that is, $\log y = n$

Then $\frac{x}{y} = 10^{m-n}$. $\therefore \log \frac{x}{y} = m - n = \log x - \log y$.

Thus, to find $37.46 \div 853.4$ we must subtract 2.9311 from 1.5736. We could say that

$$\begin{aligned} \log 37.46 &= 1.5736 \\ \log 853.4 &= \underline{2.9311} \end{aligned}$$

$$1.5736 - 2.9311 = -1.3575$$

but this produces a negative mantissa. Since the table contains only positive mantissas we would need to change -1.3575 to $.6425 - 2$. (The pupil should prove by actual subtraction that these expressions are equal.) The same result, however, can be obtained more easily, by writing 1.5736 as $11.5736 - 10$, and then subtracting in the usual way.

$$\begin{aligned} \log 37.46 &= 11.5736 - 10 \\ \log 853.4 &= \underline{2.9311} \\ & 8.6425 - 10 \end{aligned}$$

EXAMPLE 1. Divide 87.96 by .06932.

To subtract the characteristic -2, change it to $+2$ and add. The characteristic of the result is 3.

$$\begin{aligned} \log 87.96 &= 1.9443 \\ \log \underline{.06932} &= \underline{.8408 - 2} \\ \log [\quad] &= 1.1035 + 2 \end{aligned}$$

EXAMPLE 2. Divide 30.82 by .8132.

Here $\log 30.82 = 1.4889$ but this is written as $11.4889 - 10$. The characteristic of the result is $10 - 9$, or 1.

$$\begin{aligned} \log 30.82 &= 11.4889 - 10 \\ \log .8132 &= \underline{.9102 - 1} \\ \log [\quad] &= 10.5787 - 9 \end{aligned}$$

In some tables (page 277) where the characteristic must be printed, a logarithm like $.1157 - 1$ is printed 9.1157, the number -10 being understood.

275. EXERCISES

Find the values of the fractions:

1. $\dfrac{675.3}{38.64}$ 4. $\dfrac{326.6}{511.2}$ 7. $\dfrac{20.12}{692.3}$

2. $\dfrac{752.7}{20.63}$ 5. $\dfrac{45.37}{619.1}$ 8. $\dfrac{43.75}{.05121}$

3. $\dfrac{84.64}{551.3}$ 6. $\dfrac{21.68}{51.34}$ 9. $\dfrac{.05002}{681.2}$

10. $\dfrac{703.1 \times 5.684}{224.3}$

In ex. 10 do not find the actual value of the numerator. After finding its logarithm, subtract the logarithm of the denominator.

11. $\dfrac{64.32 \times .5816}{1.964}$ 13. $\dfrac{587.8}{234.2 \times 36.24}$

12. $\dfrac{392.1 \times .2972}{4.587}$ 14. $\dfrac{.3846}{.3674 \times .8122}$

276. Remarks on the Arithmetic Work. When interpolating, we often need to decide whether a number like $1\frac{1}{2}$ should be called 1 or 2. The following is a good plan:

To find the mantissa for 815.3 it is necessary to take .3 of 5 (the value of D). We call this 2 rather than 1 so that the result, .9114, will end in an *even* number. To find the mantissa for 401.5 it is necessary to take .5 of 11 (the value of D). We call this 5 rather than 6 so that the result, .6036, will end in an *even* number. Thus, in a long problem we decrease the mantissa about as often as we increase it, and thereby counterbalance errors.

When multiplying .8524 by .3872 we find that the logarithm of the product is $.5185 - 1$. The antilogarithm of $.5185 - 1$ should then be written .3300, and not .33, so as to show that a four-place table was used and that the last two figures are *known* to be zeros.

Similarly, the product 85.24×3.872 should be written 330.0 so that the product will have *four* figures in it.

277. Finding Powers.

THEOREM. *The logarithm of the pth power of a number is p times the logarithm of the number.*

That is, $\log x^p = p \log x$

Proof. Let $x = 10^m$; that is, $\log x = m$

Then $x^p = (10^m)^p = 10^{mp}$. $\therefore \log x^p = mp = p \log x$.

Notice in the examples below that the characteristic is multiplied as well as the mantissa.

EXAMPLE 1. Find 45.73^3.

$$\begin{aligned} \log 45.73 &= 1.6602 \\ &\underline{3} \\ \log 45.73^3 &= 4.9806 \\ \therefore 45.73^3 &= 95{,}620 \end{aligned}$$

EXAMPLE 2. Find $.4573^3$.

$$\begin{aligned} \log .4573 &= .6602 - 1 \\ &\underline{3} \\ \log .4573^3 &= 1.9806 - 3 \\ \therefore .4573^3 &= .09562 \end{aligned}$$

278. EXERCISES

Find the indicated powers:

1. $(65.30)^3$ ANS. 278,400

2. $(29.50)^2$
3. $(4.263)^5$
4. $(3.928)^4$
5. $(.6383)^2$
6. $(.07386)^3$
7. $(.08322)^2$
8. $(.4322)^3$
9. $(.03639)^2$
10. $(.3281)^3$

11. $(-3.682)^3$. Since a negative number has no logarithm, find $(3.682)^3$, and then state the value of $(-3.682)^3$.

12. $3.142(5.514)^2$. Arrange the work thus:

$$\begin{aligned} \log 5.514 &= \\ &\underline{} \\ \log 5.514^2 &= \\ \log 3.142 &= \\ &\underline{} \\ \log [\quad] &= \end{aligned}$$

13. $(-6.238)^4$
14. $.2234(8.135)^2$
15. $3.802(.05631)^2$
16. $5.053(-2.184)^2$
17. $\dfrac{1.803(5.974)^3}{7.471}$
18. $\dfrac{2.914 \times 6.310}{(.9316)^2}$

279. Finding Roots. Since the laws of exponents apply to all exponents, whether integral or fractional, the theorem on page 232 can be used to find any real root of a number. The power $\frac{1}{2}$, for example, gives the square root.

To find any real root of a number, divide the logarithm of the number by the index of the root and find the antilogarithm of the result.

Notice in example 2, below, how the logarithm is written so that the *negative characteristic* may be divided exactly.

EXAMPLE 1. Find $\sqrt[3]{63.73}$.

$$\log 63.73 = 1.8043$$

Divide by 3:

$$\log \sqrt[3]{63.73} = .6014$$

$$\therefore \sqrt[3]{63.73} = 3.994$$

EXAMPLE 2. Find $\sqrt[3]{.6373}$.

$$\log .6373 = .8043 - 1$$

$$\text{or } 2.8043 - 3$$

$$\log \sqrt[3]{.6373} = .9348 - 1$$

$$\therefore \sqrt[3]{.6373} = .8606$$

In the above work, when we divide 1.8043 by 3 we really get $.6014\frac{1}{3}$; but the $\frac{1}{3}$ is neglected as it is less than $\frac{1}{2}$. Also, in dividing 2.8043 by 3 we get $.9347\frac{2}{3}$; but since $\frac{2}{3}$ is more than $\frac{1}{2}$, we write the quotient as .9348. If the fraction is just $\frac{1}{2}$, we add it as 1 or drop it entirely — whichever will make the result an even number.

280. EXERCISES

1. (*a*) Find the square root, the cube root, the fourth root, and the sixth root of 763.8.

(*b*) Find the same roots of .7638.

2. Find $(70.23)^{\frac{3}{2}}$. Multiply the logarithm of 70.23 by $\frac{3}{2}$. Multiply first by 3 and then divide by 2, so that you can see how large the fractions are that you neglect.

Find the values of the following quantities:

3. $(6.825)^{\frac{3}{2}}$

4. $(-92.32)^{\frac{2}{3}}$

5. $\sqrt[4]{.006723}$

6. $\sqrt{264.3\sqrt{456.8}}$

7. $\sqrt{54.32\sqrt[3]{816.4}}$

281. EXERCISES FOR DRILL ON COMPUTATION

The answers to the odd-numbered exercises are stated at the bottom.

Whenever possible, check the position of the decimal point in the answer by making an estimate of the approximate value of the quantity. The quantity in ex. 1, for example, is about $\frac{500 \times 7}{(25)^2}$, or $\frac{3500}{625}$, or about 6.

Find the values of the following quantities:

1. $\dfrac{503.2 \times 6.783}{(24.64)^2}$
2. $\dfrac{310.8}{28.25 \times 4.601}$
3. $\dfrac{12.34}{.001897}$
4. $\dfrac{3.142(8.024)}{3}$
5. $\dfrac{.6534(.5609)^3}{.08032}$
6. $(86.36)^{\frac{2}{3}}$
7. $(-55.55)^{\frac{1}{3}}$
8. $\sqrt[5]{100}$
9. $\sqrt[10]{1000}$
10. $\sqrt[3]{.06543}$
11. $2.932\sqrt{.3688}$
12. $\sqrt[3]{\dfrac{1.967 \times .9834}{3.142}}$
13. $\sqrt{\dfrac{8.708 \times .7303}{.04138}}$
14. $\sqrt[3]{\dfrac{.02536 \times 178.4}{186.3}}$
15. $\dfrac{1}{876.6}\sqrt{\dfrac{321.7}{263.4}}$
16. $\dfrac{563.2}{394.7}\sqrt[3]{\dfrac{1}{.06374}}$
17. $\dfrac{\sqrt[3]{.2364}}{\sqrt{.7002}}$
18. $\sqrt{\dfrac{753.2}{\sqrt{.003979}}}$
19. $\dfrac{23.69}{8.756}\sqrt{\dfrac{65.29}{\sqrt{2.314}}}$
20. $\sqrt[3]{\dfrac{369.2 \times .005432}{19,370}}$

ANS. **1.** 5.624. **3.** 6503. **5.** 1.436. **7.** -3.816. **9.** 1.995. **11.** 1.781. **13.** 12.40. **15.** .001261. **17.** .7390. **19.** 17.72.

282. PROBLEMS

In problems involving π use the value $\log \pi = .4971$.

1. Find the area of a circle whose radius is 3.474 in.

The work should be arranged neatly and clearly. Thus:

Formula: $A = \pi r^2$.	Given: $r = 3.474$	$\log 3.474 = .5408$
$A = \pi(3.474)^2$		$\log 3.474^2 = 1.0816$
$A = 37.91$ sq. in.		$\log \pi = .4971$
		$\log [37.91] = 1.5787$

Consult page 282 for formulas you do not remember. In ex. 2 and in similar problems, change $A = \pi r^2$ to $r = \sqrt{\frac{A}{\pi}}$ so that the known numbers appear in the right member.

2. Find the radius of a circle whose area is 638.2 sq. in.

Taking the radius of the earth as 3959 mi.:

3. Find the area of the earth.

4. Find the length of the equator.

5. Find the volume of a sphere whose radius is 5.617 in.

6. Find the radius of a sphere whose volume is 853.2 cu. in.

7. The hypotenuse of a right triangle is 67.80 ft., and one arm is 46.70 ft. Find the other arm. If we call the hypotenuse h and the given arm a, then

$$b = \sqrt{h^2 - a^2} \text{ or } b = \sqrt{(h + a)(h - a)}$$

The sum of h and a, and the difference of h and a must be found before beginning the logarithmic work.

8. Find the hypotenuse of a right triangle if the sides are 26.38 in. and 30.07 in. ANS. 40.00 in.

9. Find the weight in tons of a cylindrical marble column given the data: length, 42.00 ft.; radius of base, 3.500 ft.; weight of marble, 168.8 lb. per cu. ft.

10. What is the approximate value of a sphere of solid gold 1 ft. in diameter if a cubic foot of gold weighs 1206 lb., each pound containing 7000 grains of gold, and if 23.22 grains are worth a dollar?

11. Find the area of a triangle whose sides are 169.2 ft., 251.4 ft., and 193.6 ft.

The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is half the perimeter. Add a, b, and c, and divide by 2. The result, s, is written *above* the values of a, b, and c so that the latter can be subtracted easily. As a check on this work, add $s-a$, $s-b$, and $s-c$. The sum should be s.

$s = 307.1$		$\log 307.1 = 2.4872$
$a = 169.2$	$s - a = 137.9$	$\log 137.9 = 2.1396$
$b = 251.4$	$s - b = 55.7$	$\log 55.7 = 1.7459$
$c = 193.6$	$s - c = 113.5$	$\log 113.5 = 2.0550$
$2s = 614.2$	*Check:* 307.1	$\log A^2 = 8.4277$
$s = 307.1$		$\log A = 4.2138$

[$A = 16{,}360$ sq. ft.]

Find the areas of the triangles for which:

12. $a = 302.4$ ft., $b = 453.7$ ft., $c = 393.9$ ft.

13. $a = 60.16$ ft., $b = 73.84$ ft., $c = 66.00$ ft.

14. $a = 87.20$ ft., $b = 59.72$ ft., $c = 92.48$ ft.

15. $a = 69.38$ in., $b = 103.2$ in., $c = 87.42$ in.

16. Using the above method, find the area of an equilateral triangle each side of which is 53.64 in.

In ex. 17 to 20 make use of the table on page 271.

17. The side, s, of a regular pentagon inscribed in a circle of radius r is $s = \frac{1}{2} r\sqrt{10 - 2\sqrt{5}}$. Find s if $r = 5.750$.

18. The apothem, h, of a regular decagon inscribed in a circle of radius r is $h = \frac{1}{4} r\sqrt{10 + 2\sqrt{5}}$. Find h if $r = 2.750$.

19. The area of a regular pentagon with side a is

$$A = \frac{1}{32} a^2(10 + 2\sqrt{5})^{\frac{3}{2}}.$$

Show that this is the same as $A = 1.721\, a^2$.

20. The area of a regular decagon with side a is

$$A = \frac{5}{8} a^2(3 + \sqrt{5})\sqrt{10 - 2\sqrt{5}}.$$

Show that this is the same as $A = 7.693\, a^2$.

283. Historical Note. The great power of modern calculation has been credited to three inventions: the Arabic notation, decimal fractions, and logarithms. In the beginning of the seventeenth century Kepler was studying the orbits of the planets; Galileo with his telescope was observing the stars; German mathematicians had made very accurate tables of the trigonometric ratios. All such work demanded more elaborate computations so that it has been said that the invention of logarithms doubled the life of astronomers by shortening their labors.

Logarithms were invented by John Napier in Scotland (1550–1617). It is a curious fact that while we now base the study of logarithms on our knowledge of exponents, Napier constructed logarithms before exponents came into use. Even his definition of a logarithm is not based on exponents but on the distances traveled by a point moving under certain conditions. Henry Briggs, a professor of geometry in London, so greatly admired a book written by Napier that he visited Napier chiefly to discuss the work. At their meeting various improvements were suggested, such as having zero for the logarithm of 1 in order that numbers larger than 1 might have a positive characteristic. Briggs then devoted all his time to the construction of a table of logarithms and in 1624 published a 14-place table of logarithms of the numbers from 1 to 20,000 and from 90,000 to 100,000. In 1628 Adrian Vlacq in Holland published the logarithms of numbers from 1 to 100,000 of which 70,000 had been computed by himself. Another mathematician, Bürgi of Switzerland, also conceived the idea of logarithms and constructed some tables, but his work was not published until after Napier's work was known.

It is interesting to note that these important events took place about the time that the colonies were being founded along the Atlantic seaboard. What event in American history do we associate with the year 1620?

(Supplementary Topic, Pages 238, 239)

PROBLEMS DEALING WITH COMPOUND INTEREST

284. Problem. If P dollars is deposited annually and i is the rate of interest, compounded annually, what is the amount, A, after n years?

The first deposit draws interest for n years and (see page 216) amounts to $P(1+i)^n$. The second deposit draws interest for $(n-1)$ years and amounts to $P(1+i)^{n-1}$. The third deposit amounts to $P(1+i)^{n-2}$. The last deposit draws interest for 1 yr., and amounts to $P(1+i)$. Arranging these amounts in reverse order, we find that the total is:

$$P(1+i)+P(1+i)^2+P(1+i)^3+\cdots+P(1+i)^n.$$

This is a geometric progression whose first term is $P(1+i)$ and whose ratio is $(1+i)$. Hence the sum is:

$$A=P(1+i)\frac{(1+i)^n-1}{(1+i)-1},\text{ or }A=\frac{P(1+i)}{i}[(1+i)^n-1].$$

The quantity $(1+i)^n$ which occurs in this formula is used so frequently in problems dealing with investments, premiums on insurance, etc. that special tables have been made showing the values of $(1+i)^n$. A part of such a table is shown on page 269.

EXAMPLE. Find A if $P=500$, $i=6\%$, $n=10$.

On page 269 we find that $(1.06)^{10}=1.7908477$. Since we are using a four-place table of logarithms, we must use only four figures. Hence we take $(1.06)^{10}-1=.7908$. Then

$$A=\frac{500\times1.06\times.7908}{.06}$$

The value of this fraction, when found by four-place logarithms, is 6983. The correct value, taken from an eight-place table, is \$6985.8213.

A series of equal payments at regular intervals is called an *annuity*. In books on the mathematical theory of investments various kinds of annuities are discussed.

285. PROBLEMS

Find A in the formula on page 238 if

1. $P = 400, n = 8, i = 6\%$

2. $P = 600, n = 10, i = 5\%$

3. Find the amount accumulated in 6 yr. if \$500 is deposited annually, interest being 5% compounded annually.

4. Find the amount on \$2000 for 12 yr. at 4% compounded annually. (If the table on page 269 is used, and logarithms are not employed in the multiplication, the amount can be found correct to eight figures.)

5. By an argument like that on page 216, show that if money is compounded semiannually the amount on P dollars after n years is $A = P\left(1 + \frac{i}{2}\right)^{2n}$ and if compounded quarterly, the amount is $A = P\left(1 + \frac{i}{4}\right)^{4n}$.

6. A debt of D dollars is to be paid in n equal annual payments of p dollars each, the interest being compounded annually at $i\%$. Find the annual payment, p.

SUGGESTIONS. At the end of the first year the debt is $D(1 + i)$, and the payment made at that time reduces the debt for the next year to $D(1 + i) - p$. The debt at the end of the second year is $(1 + i)$ times this sum, or $D(1 + i)^2 - p(1 + i)$. The payment then reduces the debt to $D(1 + i)^2 - p(1 + i) - p$, etc.

When the last payment is made, the debt must be equal to *zero*. Hence an equation can be formed, and solved for p. Show that

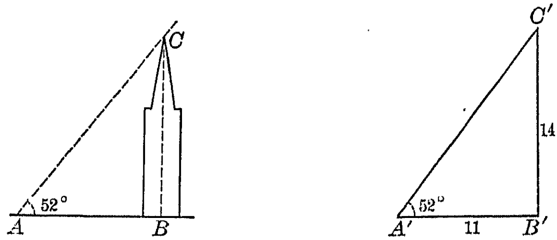
$$p = \frac{iD(1 + i)^n}{(1 + i)^n - 1}$$

7. Use the formula in ex. 6 to find what annual payment will clear a debt of \$1000 in 10 yr., interest being 6%.

CHAPTER XVIII

NUMERICAL TRIGONOMETRY

286. Problem. To find the height of the tower BC an engineer measures the angle at A and finds it to be 52°. If the distance AB is 150 ft., what is the height of the tower?



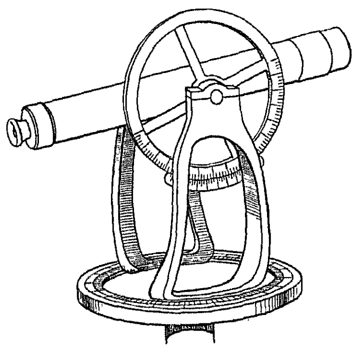
The problem can be solved by drawing a triangle similar to $\triangle ABC$, as $\triangle A'B'C'$ in the right-hand figure. Since the tower is perpendicular to the earth, $\angle A'B'C'$ is made 90°, and $\angle A'$ is made equal to $\angle A$, or 52°. The lengths of $A'B'$ and $B'C'$ are next measured. Suppose that $B'C' = 14$ in., and $A'B' = 11$ in. Then $\frac{B'C'}{A'B'} = \frac{14}{11}$, or about 1.27.

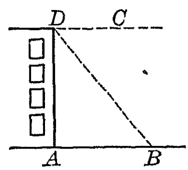
From the similar triangles we know that $\frac{BC}{AB} = \frac{B'C'}{A'B'}$. Hence $\frac{BC}{AB} = 1.27$ (approximately).

Substitute $AB = 150$ in this equation and find BC.

The accuracy of the solution depends on the accuracy of the ratio of $B'C'$ to $A'B'$. Hence these ratios have been carefully determined and put into tables.

287. An engineer measures the angle between two lines or directions by means of an instrument called a *transit.* Its essential parts are a telescope, through which the object is seen, and a horizontal and a vertical protractor. The telescope is first pointed toward one of the objects and the reading on the protractor is noted; the telescope is then turned so as to point toward the other object. The protractor then shows how many degrees the telescope was turned. Although corrections must be made in some work because the telescope is not on the ground, we shall neglect this correction in our work unless the problem asks for it.





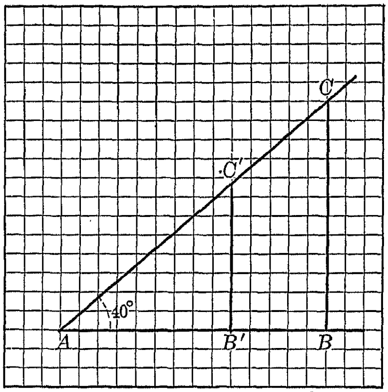
In stating a problem an engineer uses the phrases: *angle of elevation* and *angle of depression.*

In the right-hand figure above, the angle ABD is called the angle of elevation at B of the point D because this angle measures how much the telescope must be turned upwards from a horizontal position to point towards D.

The angle CDB is called the angle of depression at D of the point B because this angle measures how much the telescope must be turned downwards if the transit is at D.

Considering that DC and AB are parallel lines, what theorem of geometry states that the angle of depression at D equals the angle of elevation at B?

288. CLASS EXERCISES



1. On a sheet of squared paper draw a horizontal line, and at one end of it, using a protractor, draw a 40° angle.

At any place draw a vertical line, as BC. (Not all the pupils should draw the line BC at the same distance from A.) If squared paper is not used, make sure that angle $B = 90°$.

Measure the length BC, or count the number of squares along BC. Measure the length of AB in the same units.

Find the value of $\frac{BC}{AB}$. Does $\frac{BC}{AB} = \frac{B'C'}{AB'}$?

From your knowledge of similar triangles explain why the values of these fractions do not depend on how far from A the vertical line is drawn.

2. Each of four sections of the class should choose one of the angles 10°, 20°, 30°, 50°. Each pupil should draw on squared paper the angle his section has chosen and then complete the figure so as to have a right triangle as in ex. 1. Aim to have the triangles vary in size.

As in ex. 1, measure BC and AB, and divide BC by AB.

For comparison, the correct values are stated here:

Angle	10°	20°	30°	40°	50°
Ratio	.1763	.3640	.5774	.8391	1.1918

The pupil may be interested in showing that the ratio for a 50° angle is the reciprocal of that for a 40° angle, and for a 60° angle is the reciprocal of that for a 30° angle, etc.

289. Definition of Tangent. The ratio that was found on page 242 is called the **tangent of the angle A**, usually written **tan A.** The ratio depends only on the size of the angle, not on the size of the triangle.

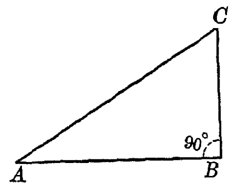
Since triangles do not always have one line in a horizontal position and one vertical, another definition is as follows:

In the right triangle ABC, we call

AC the hypotenuse,

BC the side opposite angle A,

AB the side adjacent to angle A.



The tangent of angle A equals the ratio of the opposite side to the adjacent side, or

$$\tan A = \frac{\text{the opposite side}}{\text{the adjacent side}}, \text{ or } \frac{BC}{AB}$$

290. Tables. The values of the tangent ratio for angles between 0° and 89° 50′ are stated on pages 273 to 275. The use of the words "sine" and "cosine" will be explained later.

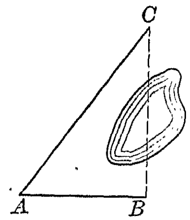
The ratios, except for certain angles, are approximate numbers; hence we need to remember the rule:

When multiplying approximate numbers, do not keep more significant figures in the product than there are in that factor which has the smaller number of significant figures.

The above rule shows that we should not measure one line of a triangle less accurately than another line, nor should we measure the angles less carefully than the lines. Since the ratios contain four figures, the lines should show four figures and the angles should be measured to the nearest hundredth of a degree (to the nearest minute usually in practice).

Hence, by a line like 26 ft. we mean 26.00 ft., the number being correct to the nearest hundredth; and by 326 we mean 326.0 so that the number will have four figures. Likewise, by an angle 52° we mean 52° 0′.

291. Problem. To find the distance from B to C, a surveyor locates the point A so that ABC will be a right triangle. Then he finds, by measuring, that



$$AB = 463.9 \text{ ft.}, \angle A = 53° 50'.$$

What is the distance BC?

From the definition of the tangent ratio, we know that

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{BC}{AB}, \text{ or } \tan 53° 50' = \frac{BC}{463.9}$$

$$\therefore BC = 463.9 \times \tan 53° 50'$$

The numerical work can be performed in three ways:

1. On page 275 we find that $\tan 53° 50' = 1.3680$

$\therefore BC = 463.9 \times 1.368$. Multiplying we find that $BC = 634.6$, only four figures being retained.

2. The value of 463.9×1.368 can be found by logarithms.

$$\begin{aligned} \log 463.9 &= 2.6664 \\ \log 1.368 &= \underline{.1361} \\ \log [634.6] &= 2.8025 \end{aligned}$$

A square bracket, [ ], is written around 634.6 to indicate that this number was found from its logarithm.

3. Some of the work in solution 2 can be avoided.

We need not find the value of tan 53° 50′ on page 275 and then find its logarithm on page 280 because the table on pages 277 to 279 tells at once *the logarithm of any tangent.* Thus, on page 279, we find that $\log \tan 53° 50' = 0.1361$.

The work should therefore appear as follows:

$$\begin{aligned} \tan A &= \frac{BC}{AB} \\ \tan 53° 50' &= \frac{BC}{463.9} \\ BC &= 463.9 \times \tan 53° 50' \\ BC &= 634.6 \end{aligned}$$

$$\begin{aligned} \log 463.9 &= 2.6664 \\ \log \tan 53° 50' &= \underline{0.1361} \\ \log [634.6] &= 2.8025 \end{aligned}$$

292. PROBLEMS — USING TANGENTS

All the triangles in these problems are right triangles.

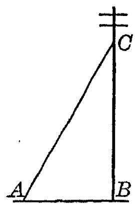
Make an accurate drawing before beginning each problem.

The tangent of an angle smaller than 45° is less than 1. Hence the logarithm has a negative characteristic. To save space in the printing of the table, log tan 32°, for example, is printed as 9.8125 instead of .8125 − 1; in other words, − 10 is understood after the logarithm of a tangent *if the angle is smaller than* 45°.

1. To find the height of a monument, the angle of elevation of the top was measured at a point 52.38 ft. from the base of the monument. If the angle of elevation is 57° 20′, what is the height of the monument?

2. Find the height of a flagstaff if the angle of elevation of the top is 42° 30′ at a point 161.5 ft. from the foot of the staff.

3. How high above the ground is the point C in the figure, if $AB = 21.75$ ft., and $\angle A = 62° 30'$?



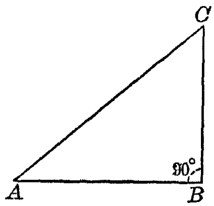
4. At a point 250.4 ft. from the base of a tower, the angle of elevation of the tower is 38° 40′. What is the height of the tower?

5. Find the height of the posts of an aerial, 60.12 ft. apart, if a line drawn from the top of one post to the base of the other makes an angle of 48° 50′ with the ground.

6. Find the height of a tent pole if, 67.25 ft. from the base, the angle of elevation of the top is 35° 20′.

7. A boy placed a transit 75 ft. 3 in. from the base of a pole and found the angle of elevation of the top to be 34° 50′. Assuming that the telescope of the transit is on the ground, what is the height of the pole? If the transit is 5 ft. high, what correction must be made?

293. Sine and Cosine of an Angle. The tangent is not the only ratio that depends entirely on the angle. If we draw an angle A, and then a right triangle, as in the figure, and measure the lengths of all three sides, we shall find that the ratios



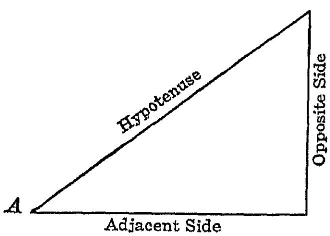
$$\frac{BC}{AC} \quad \text{and} \quad \frac{AB}{AC}$$

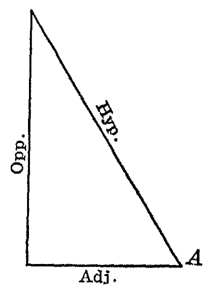
do not depend on the size of the triangle, but only on the size of the angle at A.

$\frac{BC}{AC}$ is called the **sine of angle** A, or **sin** A

$\frac{AB}{AC}$ is called the **cosine of angle** A, or **cos** A

It is best to learn the names of these ratios by using the words *opposite side*, *adjacent side*, and *hypotenuse*.





The definitions of the ratios are:

$$\text{Tangent of } A = \frac{\text{opposite side}}{\text{adjacent side}}, \quad \text{or} \quad \tan A = \frac{\text{opp.}}{\text{adj.}}$$

$$\text{Sine of } A = \frac{\text{opposite side}}{\text{hypotenuse}}, \quad \text{or} \quad \sin A = \frac{\text{opp.}}{\text{hyp.}}$$

$$\text{Cosine of } A = \frac{\text{adjacent side}}{\text{hypotenuse}}, \quad \text{or} \quad \cos A = \frac{\text{adj.}}{\text{hyp.}}$$

294. EXERCISES — USE OF THE TABLES

1. Find on pages 273 to 275 the values of:

sin 16° 40′	sin 35° 10′	sin 58° 40′	sin 86° 20′
cos 16° 40′	cos 35° 10′	cos 58° 40′	cos 70° 40′
tan 16° 40′	tan 35° 10′	tan 58° 40′	tan 72° 0′

2. Compare sin 60° with sin 30°; sin 80° with sin 40°; sin 58° 40′ with sin 29° 20′. If an angle is doubled in size, is the value of the sine of the angle doubled?

3. (*a*) Find cos 30° and cos 45°. Is the sum of these two decimals equal to cos 75°?

(*b*) Show from the table that sin 45° does not equal sin 15° added to sin 30°.

4. If an angle increases in size, does its sine become larger or smaller? its cosine? its tangent?

5. Divide sin 40° by cos 40°. Compare the quotient with tan 40°. Divide sin 30° by cos 30°. Compare the quotient with tan 30°. What discovery have you made?

6. Find the size of angle A, B, etc.:

$\sin A = .5000$	$\sin D = .6018$	$\sin G = .9315$
$\cos B = .2108$	$\cos E = .5150$	$\cos H = .9969$
$\tan C = .2401$	$\tan F = 1.0850$	$\tan J = 2.5605$

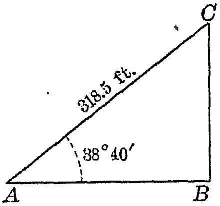
7. Since the sine and the cosine of an angle are smaller than 1, the logarithms of sines and cosines have negative characteristics. Hence the number − 10 is understood after the *logarithms of all sines and cosines* on pages 277 to 279. Find in this table:

log sin 5° 40′	log sin 44° 20′	log sin 57° 40′
log cos 5° 40′	log cos 44° 20′	log cos 57° 40′
log tan 5° 40′	log tan 44° 20′	log tan 57° 40′

8. Find, on pages 277 to 279, the size of angle A, B, etc., if

$\log \sin A = 9.6007$	$\log \tan C = 8.8960$	$\log \cos E = 9.7400$
$\log \cos B = 9.1781$	$\log \sin D = 9.8035$	$\log \tan F = 9.9975$

295. Problem. The hypotenuse, AC, of a right triangle is 318.5 ft., and $\angle A = 38° 40'$. Find BC and AB.



From the definitions,

$$\sin 38° 40' = \frac{BC}{318.5}$$

$$\cos 38° 40' = \frac{AB}{318.5}$$

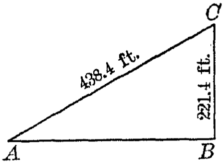
Hence $BC = 318.5 \sin 38° 40'$
$AB = 318.5 \cos 38° 40'$

The logarithmic work should then be done as follows:

$$\begin{array}{lcl} \log 318.5 & = & 2.5031 \\ \log \sin 38° 40' & = & \underline{9.7957 - 10} \\ \log\,[199.0] & = & 12.2988 - 10 \\ & & BC = 199.0 \text{ ft.} \end{array} \qquad \begin{array}{lcl} \log 318.5 & = & 2.5031 \\ \log \cos 38° 40' & = & \underline{9.8925 - 10} \\ \log\,[248.6] & = & 12.3956 - 10 \\ & & AB = 248.6 \text{ ft.} \end{array}$$

296. Finding the Angle when Two Sides are Known.

Problem. In $\triangle ABC$, $BC = 221.4$ ft., and $AC = 438.4$ ft. Find the size of $\angle A$.



From the definition,

$$\sin A = \frac{221.4}{438.4}$$

We can divide 221.4 by 438.4, obtaining .5050, and then find, on page 275, that .5050 is the sine of 30° 20′.

However, the logarithmic work is easier. Thus,

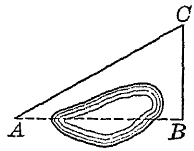
$$\begin{array}{rcll} \log 221.4 & = & 12.3452 - 10 & \\ \log 438.4 & = & \underline{2.6419} & \\ \log \sin A & = & 9.7033 - 10 & \text{(By subtraction.)} \\ \angle A & = & 30° 20' & \text{(This is found on page 279.)} \end{array}$$

If AB and AC were the given lines, the quotient would be the cosine of $\angle A$. What ratio is determined if BC and AB are the given lines?

297. PROBLEMS — USING SINES AND COSINES

1. A hill has a uniform incline of $9° 30'$. What is the height of the hill if the distance from its base to its top is 75.25 yd.?

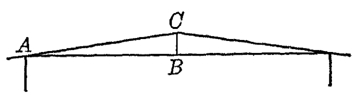
2. The point B is due east of the point A. To find the distance between them, a surveyor measures the line AC until he reaches a point, C, due north of B. Find AB if $AC = 184.2$ yd., and $\angle A = 29° 40'$.



3. In $\triangle ABC$, $AC = 714.4$ ft., $BC = 240.4$ ft., and $\angle B = 90°$. Find $\angle A$. After finding $\angle A$, use the angle and AC to find AB. (This method of finding AB is more accurate than the formula $AB^2 = AC^2 - BC^2$ because the latter contains the difference, instead of a product, of two numbers.)

4. How high on a vertical wall will a ladder 28 ft. 3 in. long reach if the foot of the ladder is 12 ft. 9 in. from the wall? (As in ex. 3, find one of the acute angles of the triangle first. Then use this angle and the hypotenuse to find the third side of the triangle.)

5. In the plan of the shed shown at the right, $AC = 50$ ft. 2 in., and $BC = 3$ ft. 6 in. Find about what angle the roof makes with the horizontal.



6. In ex. 5 what angle does the roof make with the horizontal if $AC = 41$ ft. 10 in. and the shed is 62 ft. wide?

7. The angle between two streets is $70° 30'$. If the triangular lot on the corner is 254.3 ft. long on one street and 302.5 ft. long on the other street, find the area of the lot. (First find the altitude from one corner to one of the given sides. Do not find the actual value of the altitude, as only its logarithm is needed in the work.)

298. Interpolation. To find tan 18° 34′ we find on page 274 that tan 18° 30′ = .3346 and tan 18° 40′ = .3378. From the values of tan 18° 30′ and tan 18° 40′ we can find tan 18° 34′ by *interpolation* as on page 226.

EXAMPLE 1. Find tan 18° 34′.

On page 274,
tan 18° 30′ = .3346
tan 18° 40′ = .3378

34′ is $\frac{4}{10}$ of the way from 30′ to 40′
The tabular difference is 32. This is found by subtracting 3346 from 3378
.4 × 32 = 13 (approximately)
tan 18° 34′ = .3346 + .0013 = .3359

EXAMPLE 2. Find sin 31° 16′.

On page 275,
sin 31° 10′ = .5175
sin 31° 20′ = .5200

16′ is $\frac{6}{10}$ of the way from 10′ to 20′
The tabular difference is 25
.6 × 25 = 15
sin 31° 16′ = .5175 + .0015 = .5190

EXAMPLE 3. Find cos 54° 27′.

When finding the value of a *cosine*, extra care is needed because the cosine *decreases* as the angle *increases*.

On page 275,
cos 54° 20′ = .5831
cos 54° 30′ = .5807

27′ is $\frac{7}{10}$ of the way from 20′ to 30′
The tabular difference is 24
.7 × 24 = 17 (approximately)
Subtract .0017 from .5831
cos 54° 27′ = .5831 − .0017 = .5814

EXAMPLE 4. Find log sin 35° 43′.

On page 279,
log sin 35° 40′ = 9.7657
log sin 35° 50′ = 9.7675

43′ is $\frac{3}{10}$ of the way from 40′ to 50′
The tabular difference is 18
.3 × 18 = 5 (approximately)
log sin 35° 43′ = 9.7657 + .0005 = 9.7662

EXAMPLE 5. Find log cos 65° 48′.

On page 278,
log cos 65° 40′ = 9.6149
log cos 65° 50′ = 9.6121

48′ is $\frac{8}{10}$ of the way from 40′ to 50′
The tabular difference is 28
.8 × 28 = 22 (approximately)
As in example 3, *subtract* 22
log cos 65° 48′ = 9.6149 − .0022 = 9.6127

299. EXERCISES — USE OF THE TABLES

Find the value of:

1. sin 12° 18′	5. tan 8° 25′	9. cos 6° 35′
2. sin 25° 32′	6. tan 25° 42′	10. cos 13° 23′
3. sin 43° 41′	7. tan 49° 31′	11. cos 40° 7′
4. sin 65° 38′	8. tan 63° 36′	12. cos 72° 26′

13. log sin 17° 25′	19. log tan 51° 7′
14. log sin 17° 26′	20. log tan 74° 46′
15. log sin 34° 42′	21. log cos 16° 53′
16. log sin 60° 28′	22. log cos 23° 4′
17. log tan 15° 13′	23. log cos 34° 37′
18. log tan 31° 48′	24. log cos 69° 32′

25. Find angle A if $\sin A = .6631$.

On page 275 we see that the angle is between 41° 30′ and 41° 40′, but we can give a more exact answer. We notice that

$$\left.\begin{array}{l}\left.\begin{array}{l}\sin 41^\circ\, 30' = .6626\\ \sin A \qquad\; = .6631\end{array}\right\}\begin{array}{c}\text{difference}\\ \text{is } \mathbf{5}\end{array}\\ \sin 41^\circ\, 40' = .6648\end{array}\right\}\begin{array}{c}\text{difference}\\ \text{is } \mathbf{22}\end{array}$$

Hence A is 41° 30′ + $\frac{5}{22}$ of 10′, or 41° 30′ + 2′ (approx.), or 41° 32′.

To find a fractional part of 10′ use the suggestion on page 228; that is, annex a 0 to the numerator and then divide.

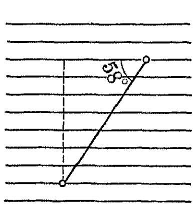
Find the angle A to the nearest minute if:

26. $\sin A = .3434$	30. $\tan A = .2041$	34. $\cos A = .5070$
27. $\sin A = .5030$	31. $\tan A = .4047$	35. $\cos A = .7046$
28. $\sin A = .8847$	32. $\tan A = .9036$	36. $\cos A = .8239$
29. $\sin A = .9855$	33. $\tan A = .9602$	37. $\cos A = .9934$

38. $\log \sin A = 9.6815$	44. $\log \tan A = 9.9782$
39. $\log \sin A = 9.7409$	45. $\log \tan A = 0.2509$
40. $\log \sin A = 9.7714$	46. $\log \cos A = 9.9715$
41. $\log \sin A = 9.9013$	47. $\log \cos A = 9.9527$
42. $\log \tan A = 9.4336$	48. $\log \cos A = 9.8924$
43. $\log \tan A = 9.6627$	49. $\log \cos A = 9.7633$

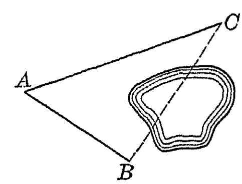
300. PROBLEMS

1. To find the height of a smokestack an engineer selects a point 260.5 ft. from the base and finds that the angle of elevation of the top is 39° 12′. What is the height of the smokestack?

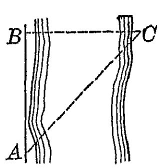


2. In a football game, a player ran from the 45-yard line to the 10-yard line, as shown in the figure. If his path made an angle of 58° with the 5-yard lines, how many yards did he run?

3. A troop of boy scouts found the distance BC across a lake by marking off AB at right angles to BC, measuring $AB = 68.75$ yd. and $\angle A = 52° 32'$. What was the distance from B to C?



4. To find the width of a river, a line BA is laid off along the bank so that $\angle B = 90°$. If $AB = 623.1$ ft. and $\angle A = 42° 16'$, what is the width of the river?



5. When the angle of elevation of the sun is 42°, how long a shadow will a 50-foot flagstaff cast on level ground?

6. In ex. 5 one angle of the right triangle is 42°. What must be the size of the other acute angle? Work ex. 5 again, using this new angle as if it were the only angle given in the problem.

Is one solution easier than the other? If addition is easier than subtraction, then, when using tangents, select carefully the better angle with which to work.

7. From the top of a lighthouse 142.3 ft. above sea level the angle of depression of a boat is 28° 36′. Find the distance from the boat to the lighthouse.

8. An observer in a balloon knows that he is 3025 ft. directly above the town of Borland. He finds the angle of depression of the next town to be $24° 50'$. What is the distance between the towns?

9. From one bank of a river the angle of elevation of the top of a cliff on the other side is $33° 19'$. If the cliff is 120.2 ft. high, how wide is the river?

10. From the top of a cliff 240.4 ft. above sea level the angle of depression of a boat is seen to be $20° 17'$. How far is the boat from the foot of the cliff?

11. (*a*) The Washington Monument is 555 ft. high. Find its angle of elevation at a point 1 mi. away.

(*b*) Find the angle of elevation at a point $\frac{1}{2}$ mi. away. Is one angle double the other?

12. If at a certain time the top of the Statue of Liberty in New York Harbor is 300 ft. above sea level, and a boat finds that the angle of elevation of the top is $30° 25'$, how far is the boat from the statue?

13. In some places the slope of the railroad that goes up Pikes Peak is 18%; that is, it rises 18 ft. vertically for each 100 ft. measured horizontally. Find what angle has a tangent of .1800.

14. The angle of elevation of the top of a building is $55° 28'$ at a point 156.4 ft. from the building. Find the height of the building.

15. A tree that is known to be 70 ft. high stands on the edge of a swamp. At a point, A, across the swamp, the angle of elevation of the top of the tree is $16° 13'$. What is the width of the swamp from the point A to the tree?

16. At a certain point, A, the angle of elevation of a balloon is $28° 16'$. The point D, directly below the balloon, is 351.4 yd. from A. How high in the air is the balloon?

301. MISCELLANEOUS EXERCISES

In these exercises $AB = c$, $BC = a$, $AC = b$, $\angle B = 90°$.

1. Given $a = 432.7$, $c = 267.3$. Find the other parts of the triangle. Ans. $b = 508.6$, $\angle A = 58° \, 18'$, $\angle C = 31° \, 42'$.

If the value of b is found as in ex. 3, page 249, we can check the work now by using $a = \sqrt{(b + c)(b - c)}$. In the following check, $(b + c)$ is written *above* b and c in order to leave room *below* b and c for the value of $(b - c)$.

$b + c = 775.9$	$\log 775.9 = 2.8898$
$b = 508.6$	$\log 241.3 = 2.3825$
$c = 267.3$	$\log a^2 = 5.2723$
$b - c = 241.3$	$\log [432.7] = 2.6362$

This checks the work because a was given as 432.7. Sometimes there is a difference of 1 or 2 in the last figure.

Find the missing lines and angles below:

	a	b	c	A	C
2.		83.62			12° 21′
3.		5.458			57° 18′
4.			374.5	32° 43′	
5.			54.86	43° 26′	
6.			436.7	37° 54′	
7.	66.34			48° 47′	
8.	75.82				54° 6′
9.	69.43				33° 12′
10.	142.9	226.8			
11.	12.18	19.73			
12.		32.63	21.45		
13.		4.602	1.520		
14.	26.83		41.36		
15.	21.15		40.32		

302. Historical Note. Hipparchus, a Greek astronomer who lived in the second century B.C., is credited with the invention of trigonometry. Although all his works are lost, historians believe he was the first to compute tables resembling our tables of sines. Hipparchus was interested in trigonometry because the latter was necessary in his astronomical work; and for a long time trigonometry was looked upon more as an aid in the study of spherical triangles (triangles on a sphere) than as an aid in engineering or surveying.

Among the famous astronomers of the tenth century A.D. was an Arabian, Al Battani. A Latin translation of his work introduced the word "sinus" as the name of one of the ratios.

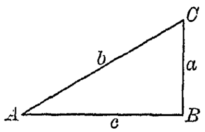
The first work in the making of accurate tables of the ratios was done in the fifteenth century by German astronomers, of which the greatest was Regiomontanus (1436–1476). He traveled much in Europe in order to learn what the Greeks and Arabs had written on mathematics, and then wrote a textbook on trigonometry.

The work of Napier (page 237) on logarithms greatly lessened the arithmetic computations in astronomy. While trigonometry was at first of interest only to astronomers and was used chiefly in the solution of triangles, the subject has since developed so that every field of mathematics has use for it. Thus, one of the formulas for finding the roots of a cubic equation involves the sines of certain angles. The various parts of mathematics now overlap and every discovery in one field is useful in other places. We may think there is little relation between the number π, the logarithms of numbers, and such a symbol as i for $\sqrt{-1}$, but the equation $\pi = 2\,i \log_e \frac{1-i}{1+i}$, discovered by Fagnano, an Italian, shows how closely related are the various parts of mathematics.

(Supplementary Topics, Pages 256, 257)

303. EXERCISES

1. In the right triangle ABC, we know that $c^2 + a^2 = b^2$. If each term of this equation is divided by b^2, then



$$\left(\frac{c}{b}\right)^2 + \left(\frac{a}{b}\right)^2 = 1$$

What ratio is given by $\frac{c}{b}$?

What ratio is given by $\frac{a}{b}$?

Show that the above equation states that for any angle:

$$\cos^2 A + \sin^2 A = 1$$

2. Verify the relation in ex. 1 by squaring the sine and the cosine of some angle.

3. Tan $A = \frac{a}{c}$, and this can be written as $\frac{a}{b} \div \frac{c}{b}$. From this expression show that the tangent of any angle equals its sine divided by its cosine. Verify the statement by using the values for some angle.

4. The ratio $\frac{c}{b}$ is what ratio for the angle A? for angle C? Prove that the cosine of any angle is the sine of its complementary angle.

5. Besides the three trigonometric ratios studied in this chapter there are three others defined as follows:

cotangent $A = \frac{c}{a}$ cosecant $A = \frac{b}{a}$ secant $A = \frac{b}{c}$

The respective abbreviations are: cot A, cosec A, and sec A.

State the definitions using the words *opposite side*, *adjacent side*, and *hypotenuse*.

Prove the following relations:

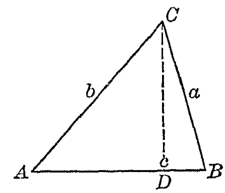
6. $\sin A \cdot \operatorname{cosec} A = 1$

7. $\cos A \cdot \sec A = 1$

8. $\tan A \cdot \cot A = 1$

9. $\tan A = \cot(90 - A)$

304. The Area of a Triangle. In the figure below, the line DC is the altitude from C.



If K denotes the area of $\triangle ABC$,

then $\quad K = \frac{1}{2} c \cdot DC.$

But $DC = b \sin A.$

Hence the area of any triangle is

$$\tfrac{1}{2} bc \sin A$$

where A is the angle between b and c.

EXAMPLE. Find the area of a triangle if two sides are 253.4 ft. and 432.1 ft., and the included angle is 38° 43′.

Given: $b = 253.4$
$c = 432.1$
$A = 38^\circ 43'$
Formula: $K = \frac{1}{2} bc \sin A$
The area is 34,250 sq. ft.

log 253.4	= 2.4038
log 432.1	= 2.6356
log sin 38° 43′	= 9.7962
	4.8356
log 2	= .3010
log [34,250]	= 4.5346

305. EXERCISES

Find the areas of the following triangles:

1. $b = 359.1$ ft., $c = 98.35$ ft., $A = 52^\circ 29'$
2. $b = 54.04$ ft., $c = 45.02$ ft., $A = 21^\circ 32'$
3. $b = 31.87$ ft., $c = 42.66$ ft., $A = 26^\circ 44'$
4. $b = 6.732$ ft., $c = 80.46$ ft., $A = 59^\circ 56'$
5. $b = 263.1$ ft., $c = 325.6$ ft., $A = 32^\circ 13'$
6. $a = 72.52$ yd., $b = 63.47$ yd., $C = 43^\circ 57'$
7. $a = 232.7$ rd., $b = 124.6$ rd., $C = 78^\circ 21'$
8. $a = 64.76$ rd., $c = 583.6$ rd., $B = 35^\circ 36'$
9. $a = 43.75$ rd., $c = 55.23$ rd., $B = 63^\circ 42'$

10. Prove that the area of a parallelogram equals the product of two adjacent sides and the sine of the angle between them.

EXERCISES FOR A GENERAL REVIEW

306. FACTORING — FRACTIONS

Find the prime factors of the following:

1. $x^4 - 2x^2(a^2 + b^2) + (a^2 - b^2)^2$
2. $(a + b)^2 - (c + d)^2 + (a + c)^2 - (b + d)^2$
3. $y^2 - 2y(r + s) - rs(r - 2)(s + 2)$
4. $x^4 + 2x^3y + x^2y^2 - 1$
5. $P(1 + i) + iP(1 + i)$

Simplify the following:

6. $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \dfrac{n(n-1)}{1 \cdot 2}$
7. $\dfrac{a^4 - x^4}{a^3 - x^3} \div \dfrac{a^2 + x^2}{a - x}$
8. $\dfrac{6a(2a + 1) - a(a + 1)(a + 8)}{6a - (a + 1)(a + 2)}$
9. $P(1 + i)^{20} \div \dfrac{(1 + i)^{10} - 1}{(1 + i) - 1}$
10. $\left(\dfrac{x + 4}{x^2 - 9} + \dfrac{1}{2 - x}\right) \div \dfrac{2x + 1}{x^2 - 5x + 6}$

11. $\dfrac{\dfrac{x - y}{x + y} + \dfrac{y}{x}}{\dfrac{x + y}{x - y} - \dfrac{x}{y}}$ 12. $\dfrac{(x^3 - y^3)\left(\dfrac{1}{x} + \dfrac{1}{y}\right)}{(x + y)\left(\dfrac{1}{x} - \dfrac{1}{y}\right)}$

13. Simplify the following solutions of ex. 10, page 113:

$$x = \frac{2a(a^2 - 1) - 4a(a - 1)}{(a + 1)(a^2 - 1) + (a - 1)(a^2 - 1)}$$

$$y = \frac{4a(a + 1) + 2a(a^2 - 1)}{(a + 1)(a^2 - 1) + (a - 1)(a^2 - 1)}$$

14. Simplify the following fraction, which arises in a certain problem in trigonometry: $\dfrac{t + \dfrac{2t^2}{1 - t^2}}{1 - t\dfrac{2t^2}{1 - t^2}}$

307. EQUATIONS

Solve the following equations and sets of equations:

1. $(2x+3)(x+5) = 2(x+3)^2 - (x-1)^2$
2. $(x^2 - 3x)^2 - 14(x^2 - 3x) + 40 = 0$
3. $(x+a)(x-b) - (x-a)(x+b) = (a-b)^2$
4. $\sqrt{x+4} + \sqrt{x-1} = 5$
5. $\dfrac{4x-1}{5} - \left(x - 1 + \dfrac{2-x}{3}\right) = 4$
6. $\dfrac{x-2}{x} - \dfrac{x-12}{x+6} = \dfrac{5}{6}$
7. $2^x 4^{x+2} = 8^{2x-1} 4^{1+x}$
8. $x^6 - 7x^3 + 10 = 0$
9. $\begin{cases} y + 3x = 9 \\ xy = 6 \end{cases}$
10. $\begin{cases} 3y + x = 10 \\ 3y^2 + 7x = 34 \end{cases}$
11. $\begin{cases} y^2 = 4(x-3) \\ 3(x-2) + y = 8 \end{cases}$
12. $\begin{cases} x^2 + 2x + 2y = -2 \\ x + y + 1 = 0 \end{cases}$
13. $\begin{cases} 2x + 3y + z = 0 \\ x + 6y + 5z = 1 \\ 4x - 3y - 2z = 3 \end{cases}$
14. $\begin{cases} 2x - 3y - 6z = 1 \\ x + y - z = 9 \\ 5x - 2y - 2z = 4 \end{cases}$
15. $\begin{cases} ax + by - bz = a^2 - b^2 \\ bx - ay + az = (a+b)^2 \\ ax - by - bz = (a-b)^2 \end{cases}$
16. $x(\sqrt{2} - 1) + \sqrt{3} = x(\sqrt{3} + \sqrt{2}) - 1 - \sqrt{2}(1 + \sqrt{3})$
17. Express A in terms of C if $A = \pi r^2$ and $C = 2\pi r$.
18. Eliminate t from the equations
$$s = \tfrac{1}{2}gt^2 + kt \text{ and } v = gt + k'$$
19. Eliminate h from $V = \pi r^2 h$ and $h = r + 2$.
20. Without solving the equation, determine the nature of the roots of the equation $3x^2 - 5x - 8 = 0$.
21. Without finding the roots, state the sum and the product of the roots of $hx^2 + (h+k)x + k^2 = 0$.

308. VARIATION — FUNCTIONS

1. Write as an equation the statement: The intensity of a light at any point varies inversely as the square of the distance of the point from the light.

2. Write as an equation: The distance a body falls from a position of rest varies as the square of the number of seconds it falls.

3. (*a*) Write as an equation: The volume of a sphere is a function of the radius.

(*b*) Write as a proportion: The volumes of two spheres are proportional to the cubes of their radii.

4. (*a*) Write as an equation: The area of a circle is a function of the square of its radius.

(*b*) Write as a proportion: The areas of two circles are proportional to the squares of their radii.

5. The statement " z varies *jointly* as x and y " means that z varies as x when y is fixed in value, and that z varies as y when x is fixed in value. Hence the relation is written: $z = kxy$, where k is some constant.

The area of a triangle varies jointly as the base and the altitude. Hence $A = kab$. What is the value of k in this equation? Show that if $a = 10$, then the area varies as the base. Show that if $b = 6$, then the area varies as the altitude.

6. If $y = \frac{1}{x}$, state whether the function (that is, y) increases or decreases as x increases.

State whether the following functions of x increase or decrease as x increases (x being positive):

7. $y = \dfrac{6}{x+2}$

8. $y = 2 + \dfrac{6}{x}$

9. $y = \dfrac{1 + \frac{2}{x}}{3}$

10. $y = \dfrac{2}{3 + \frac{1}{x}}$

309. GRAPHS

Draw the graphs from $x = -3$ to $x = 3$ of:

1. $y = x^3 - 2x + 1$ 2. $y = x^3 - x^2 - 4$

3. How can the roots of the equation $x^3 - 2x + 1 = 0$ be found from the graph drawn in ex. 1?

4. Solve graphically $2x^3 + 3x^2 - 9x - 10 = 0$.

5. Solve graphically: $\begin{cases} x^2 + y = 7 \\ x + y^2 = 11 \end{cases}$

If we try to solve this set by the substitution method, we are led to the equation $x^4 - 14x^2 + x + 38 = 0$. One root of this equation is $x = 2$; hence the left member is exactly divisible by $x - 2$. Dividing, we get the equation $x^3 + 2x^2 - 10x - 19 = 0$. The methods of solving equations of the third degree to any desired accuracy are studied in college algebra. The solutions, to two decimals, of the given set are:

$\begin{cases} x = 2 \\ y = 3 \end{cases}$ $\begin{cases} x = 3.13 \\ y = -2.81 \end{cases}$ $\begin{cases} x = -1.85 \\ y = 3.58 \end{cases}$ $\begin{cases} x = -3.28 \\ y = -3.78 \end{cases}$

6. Draw the graph of $x^2 + y^2 = 9$, and on the same axes draw the graphs of (*a*) $y = x + 1$; (*b*) $y = x + 5$.

The first line intersects the curve in two points, and the second line does not intersect the curve at all. To find the equation of a line that intersects the curve in just one point (that is, a *tangent* line), we solve the two equations: $x^2 + y^2 = 9$ and $y = x + k$. This gives the equation $2x^2 + 2kx + k^2 - 9 = 0$. For what value of k will the roots of this equation be equal? After finding k, substitute its value in $y = x + k$ and draw the graph.

As in ex. 6, find the value of k such that:

7. $y = 4x + k$ will be tangent to $y = 3x^2$.

8. $y = kx - 2$ will be tangent to $y = x^2$.

9. $y = kx - 3$ will be tangent to $y = x^2 + 2x$.

Such problems are studied further in analytic geometry.

310. RADICALS — EXPONENTS

1. Show that $1+\sqrt{\frac{5}{8}}$ equals $\frac{1}{4}(4+\sqrt{10})$.

2. By squaring the two numbers, show that $\sqrt{2+\sqrt{3}}$ and $\frac{1}{2}(\sqrt{2}+\sqrt{6})$ are the same number.

3. Prove, by checking, that $x = 1$ is a solution of

$$(x+\sqrt{3})^2+(x-\sqrt{3})^2+(x+\sqrt{3})(x-\sqrt{3})=6\,x$$

4. By solving each equation, show that the roots of $mx^2 + nx + p = 0$ are twice as large as the roots of the equation $4\,mx^2 + 2\,nx + p = 0$.

Write in the simplest form:

5. $\sqrt{.4}+\sqrt{2.5}+\sqrt{.1}$

6. $\sqrt{24}-\sqrt{\frac{1}{6}}+\sqrt{\frac{2}{3}}$

7. $\dfrac{x^{\frac{2}{3}}y}{z^{\frac{1}{3}}}\cdot\dfrac{z^{-\frac{2}{3}}}{x^{\frac{1}{3}}y^{-\frac{1}{3}}}$

8. $(4\,x^{-8}y^2)^{-\frac{3}{2}}$

9. $(8\,x^6y^{-9})^{\frac{2}{3}}$

10. $\dfrac{x^{-1}y}{z^{\frac{1}{2}}}\div\dfrac{x^{-2}y^2}{z^{\frac{2}{3}}}$

11. $\dfrac{2}{\sqrt{5}+\sqrt{3}}-\dfrac{3}{\sqrt{5}-\sqrt{2}}+\dfrac{1}{\sqrt{3}-\sqrt{2}}$

12. $\sqrt{\dfrac{50\,a}{b}}+\sqrt{\dfrac{a}{2\,b}}-\sqrt{\dfrac{2\,b}{a}}-\sqrt{\dfrac{1}{50\,ab}}$

13. $x^{\frac{2}{3}}(\sqrt[3]{x}+1)-\sqrt[3]{x^2}(x^{\frac{1}{3}}-1)$

14. $x^{\frac{1}{3}}y(x^{\frac{2}{3}}-y)-xy^{\frac{1}{3}}(x+y^{\frac{2}{3}})$

State the values of:

15. $8^{\frac{2}{3}}$

16. $8^{-\frac{1}{3}}$

17. $(8^{\frac{2}{3}})^{-3}$

18. $(8^{-1})^{\frac{2}{3}}$

19. $4^{-\frac{1}{2}}$

20. $(\frac{1}{4})^{\frac{1}{2}}$

21. $(\frac{1}{4})^{-3}$

22. 4^{-3}

23. 6^0

24. $25^{-\frac{1}{2}}$

25. $(9^{\frac{3}{2}})^{-1}$

26. $(9^{-1})^{\frac{3}{2}}$

Write with positive exponents and simplify:

27. $\dfrac{a}{a^{-1}+b^{-1}}$

28. $\dfrac{b^{-1}}{a+b^{-1}}$

29. $\dfrac{a^{-2}}{a^{-2}-b^2}$

311. PROBLEMS

1. At an entertainment the children were admitted for 15¢ and the adults for 25¢. If \$13.75 was collected and 75 people were present, how many children and how many adults were present?

2. The sides of a triangle are 9 in., 11 in., and 12 in. Find, to the nearest hundredth of an inch, the length of the altitude to the longest side of the triangle.

3. A dairyman combines milk containing 4% butter fat with some cream containing 24% butter fat. How many gallons of each shall he use to make 760 gal. containing 18% butter fat?

4. The load that a beam can safely carry varies as the square of the depth of the beam. If a beam 6 in. deep can carry 2700 lb., how much can an 8-inch beam carry?

5. A farmer has enough fencing to inclose a rectangular field whose length is 4 ft. more than 3 times its width. If, however, he builds a square inclosure with the same perimeter as the rectangle, the area of the field will be increased 64 sq. ft. Find the length and the width of the field.

Would the *method* of solving this problem be different if the question were: How many feet of fencing has the farmer?

6. If an object is thrown vertically upward with a velocity of 96 ft. a second, its distance above ground after t seconds is found by the formula $d = 96t - 16t^2$.

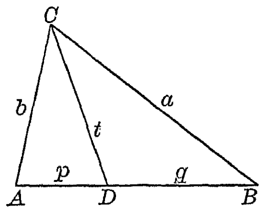
(*a*) After how many seconds does $d = 140$? Explain why both solutions of the equation have a meaning.

(*b*) Solve the equation when $d = 150$. Can the object ever reach a height of 150 ft.?

(*c*) To find the greatest height the object can reach, draw the graph of $d = 96t - 16t^2$ from $t = 0$ to $t = 6$. What is your conclusion?

312. PROBLEMS FROM GEOMETRY

1. In the figure, CD is the bisector of angle C. Hence we know that $\frac{b}{a} = \frac{p}{q}$.



From this equation and the fact that $p + q = c$, show that

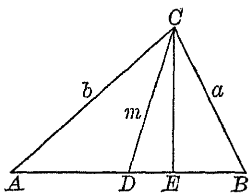
$$p = \frac{bc}{a+b}$$

and

$$q = \frac{ac}{a+b}$$

2. In ex. 1, it is also known that $t^2 = ab - pq$. From this fact, show that $t^2 = \frac{ab}{(a+b)^2}(a+b+c)(a+b-c)$.

3. If $a + b + c = 2\,s$, prove that $(a+b)^2 - c^2 = 4\,s(s-c)$.



4. In the figure, CD is a median and CE an altitude. Call $DE = x$ and $EC = y$. Note that $AD = DB = \frac{1}{2}\,c$. Write all the equations that can be found from the right triangles, eliminate x and y, and prove that

$$2\,a^2 + 2\,b^2 = c^2 + 4\,m^2$$

5. The radius of one circle is r and that of another circle is $r + a$. Find the difference in their circumferences. Does the difference depend on the value of r?

6. The circumference of one circle is C and that of another circle is $C + a$. Find the difference in their radii. Does the difference of the radii depend on the value of C?

7. The circumference of the earth is about 25,000 mi., and that of the moon is about 6800 mi. If each circumference were increased by 1 mi., would the change in the radius of the earth be more or less than the change in the radius of the moon?

313. PROGRESSIONS

1. Each stroke of an air pump removes one fourth of the air in a vessel. What fractional part of the air has been removed after 5 strokes?

2. The machinery in a factory cost $50,000 and each year it is worth 15% less than in the previous year. What is the machinery worth at the end of the fifth year?

3. A vessel containing an 80% sugar solution was emptied of one half of its contents and then filled with water. If this process was done 6 times, what was the per cent of sugar in the final solution?

4. If letters are written to each of 2 people and each of these writes to 2 others, etc., for 10 rounds, how many letters are written in all?

5. Suppose in ex. 4 that each person writes 3 letters instead of 2. Will the number of letters be greatly increased? Guess at an answer, and then find how many letters will be written.

6. On shipboard "eight bells" is rung at noon and every 4 hours thereafter. If 1 bell is rung at 12.30 P.M., 2 bells at 1 P.M., 3 at 1.30 P.M. etc., up to "eight bells" at 4 o'clock, how many bells are rung from noon till midnight?

7. The successive multiples of a number h (including the number itself) are h, $2h$, $3h$, etc. Find a formula for the sum of the first n terms of this progression.

8. If each term of a geometric progression is multiplied by k, do the new terms form a geometric progression?

9. Answer ex. 8 for an arithmetic progression.

10. Repeat ex. 8 and 9, assuming that the number k is added to each term of the progression.

11. Prove that in a geometric progression:

(a) $t_n t_m = a^2 r^{n+m-2}$

(b) If $a = r$, then $t_n t_m = r^{n+m}$

314. LOGARITHMS

Without using the tables, prove that

1. $\log \frac{5}{6} + \log 8 - \log \frac{4}{3} = \log 5$

SUGGESTION. $\log \frac{5}{6} + \log 8 - \log \frac{4}{3} = \log (\frac{5}{6} \times 8 \div \frac{4}{3})$
$= \log (\frac{5}{6} \times 8 \times \frac{3}{4}) = \log (?)$

2. $\log \frac{3}{4} - 5 \log 2 + \log 128 = \log 3$

SUGGESTION. Note that $5 \log 2 = \log 2^5 = \log 32$.

3. $\log \frac{7}{8} - \log \frac{21}{4} + 3 \log 3 - \log 5 = \log .9$

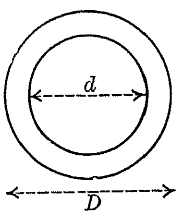
4. $\log \frac{9}{16} + 6 \log 2 - 3 \log 5 + \log 50 - \log 6 = \log 2.4$

5. $\log \sqrt{a^2 - x^2} = \frac{1}{2} \log (a + x) + \frac{1}{2} \log (a - x)$

In ex. 6 and 7 the answers should have 4 significant figures.

Find the value of:

6. $\dfrac{12}{\sqrt{4 - \pi}}$ 7. $\dfrac{30}{\sqrt{8 - \pi}}(1 + \sqrt{2})$



8. The volume of a circular tube, such as an automobile tire, is

$$V = \frac{\pi^2(D - d)^2(D + d)}{32}$$

where D and d are the largest and smallest diameters, as shown in the figure.

Find V if $D = 34$ in. and $d = 26$ in.

9. The part of a sphere between two parallel planes is called a spherical segment. If h is the distance between the planes, and R and r are the radii of the two bases of the segment, the volume of the segment is

$$V = \frac{\pi h}{2}\left[R^2 + r^2 + \frac{h^2}{3}\right]$$

Find V if $h = 2.7$ in., $R = 4.1$ in., and $r = 2.3$ in.

10. Write the formula for the capacity, G, in gallons (1 gal. = 231 cu. in.) of a cylindrical tank l feet long, the radius of the end being r feet. Find the capacity if the length is 18 ft. 3 in. and the radius is 2 ft. $4\frac{1}{2}$ in.

11. About 300 years ago the Dutch paid the Indians \$24 for the island of Manhattan. To what will \$24 amount in 300 years at 5% compounded annually (to the nearest \$100,000)?

12. The radius, r, of a circle inscribed in a triangle whose sides are a, b, and c, is

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad \text{where } s = \tfrac{1}{2}(a+b+c).$$

Find the radius of a circle inscribed in a triangle whose sides are 4.8 in., 4.2 in., and 5.4 in.

13. The radius, R, of a circle circumscribed about a triangle whose sides are a, b, and c, is

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{where } s = \tfrac{1}{2}(a+b+c)$$

Find the radius of a circle circumscribed about a triangle whose sides are 4.8 in., 4.2 in., and 5.4 in.

14. The elevation, e, in inches of the outer rail of a railroad curve of radius r feet is $e = \dfrac{dv^2}{2.683\,r}$, where d is the distance in feet between the rails, and v is the maximum speed in feet per second of the trains. Find e when $d = 4.5$, $r = 2000$, and $v = 66$.

315. BINOMIAL THEOREM

Write the first four terms of the expansions of:

1. $(a + 2x)^7$ **3.** $(x - 3y)^6$ **5.** $(x^2 - 3y)^8$

2. $(a^2 - y)^6$ **4.** $(x + 2y)^{10}$ **6.** $(x^3 + y^{\frac{1}{2}})^6$

7. Write the 4th term of $(2x - 3y)^6$.

8. Write the 5th term of $(x + \frac{1}{2})^8$.

9. Write the 6th term of $(2a - b)^7$.

Using the approximation formulas on page 218, find to the nearest thousandth:

10. $17^{\frac{3}{2}}$ **11.** $30^{\frac{1}{3}}$ **12.** $10^{\frac{1}{2}}$

316. TRIGONOMETRIC RATIOS

1. Draw a triangle whose sides are 1 in., 1 in., and $\sqrt{2}$ in. Why is this a right triangle? Which angle is the right angle? From the figure show that

$$\sin 45° = \tfrac{1}{2}\sqrt{2} \qquad \cos 45° = \tfrac{1}{2}\sqrt{2} \qquad \tan 45° = 1$$

2. The trigonometric ratios of 30° and 60° can be found in radical form by drawing an equilateral triangle. Let each side equal 2 in. Draw the altitude from one vertex. What is the length of the altitude? Why does the altitude bisect one angle? From the figure, find the sine, cosine, and tangent of 30° and also of 60°.

3. Complete the statement:

In a 60–30 right triangle, the hypotenuse equals the longer side multiplied by . . .; and the longer side equals the shorter side multiplied by . . .

4. Complete the sentence: The trigonometric ratios are functions of . . . of a triangle because their values depend on . . .

5. A carpenter's rule for constructing an angle of 22° 30′ is to construct a right triangle whose arms are 5 in. and 12 in. The angle opposite the 5 in. side is about 22° 30′. Find the size of the angle to the nearest minute.

6. Find the area of a triangle if two sides are 294.6 ft. and 318.3 ft., and the included angle is 33° 23′.

7. Find the area of a parallelogram if two adjacent sides are 576.1 ft. and 219.3 ft., and the angle between them is 49° 27′.

8. The hypotenuse of a right triangle is $m^2 + n^2$; one of the other sides is $2\,mn$. Find the third side. Write the expressions for the sine, the cosine, and the tangent of each of the acute angles in the triangle.

TABLES

VALUES OF $(1+i)^n$

n	2%	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	n
1	1.020 0000	1.025 0000	1.030 0000	1.035 0000	**1**
2	1.040 4000	1.050 6250	1.060 9000	1.071 2250	**2**
3	1.061 2080	1.076 8906	1.092 7270	1.108 7179	**3**
4	1.082 4322	1.103 8129	1.125 5088	1.147 5230	**4**
5	1.104 0808	1.131 4082	1.159 2741	1.187 6863	**5**
6	1.126 1624	1.159 6934	1.194 0523	1.229 2553	**6**
7	1.148 6857	1.188 6858	1.229 8739	1.272 2793	**7**
8	1.171 6594	1.218 4029	1.266 7701	1.316 8090	**8**
9	1.195 0926	1.248 8630	1.304 7732	1.362 8974	**9**
10	1.218 9944	1.280 0845	1.343 9164	1.410 5988	**10**
11	1.243 3743	1.312 0867	1.384 2339	1.459 9697	**11**
12	1.268 2418	1.344 8888	1.425 7609	1.511 0687	**12**
13	1.293 6066	1.378 5110	1.468 5337	1.563 9561	**13**
14	1.319 4788	1.412 9738	1.512 5897	1.618 6945	**14**
15	1.345 8683	1.448 2982	1.557 9674	1.675 3488	**15**
16	1.372 7857	1.484 5056	1.604 7064	1.733 9860	**16**
17	1.400 2414	1.521 6183	1.652 8476	1.794 6756	**17**
18	1.428 2463	1.559 6587	1.702 4331	1.857 4892	**18**
19	1.456 8112	1.598 6502	1.753 5061	1.922 5013	**19**
20	1.485 9474	1.638 6164	1.806 1112	1.989 7889	**20**

n	4%	$4\frac{1}{2}$%	5%	6%	n
1	1.040 0000	1.045 0000	1.050 0000	1.060 0000	**1**
2	1.081 6000	1.092 0250	1.102 5000	1.123 6000	**2**
3	1.124 8640	1.141 1661	1.157 6250	1.191 0160	**3**
4	1.169 8586	1.192 5186	1.215 5063	1.262 4770	**4**
5	1.216 6529	1.246 1819	1.276 2816	1.338 2256	**5**
6	1.265 3190	1.302 2601	1.340 0956	1.418 5191	**6**
7	1.315 9318	1.360 8618	1.407 1004	1.503 6303	**7**
8	1.368 5691	1.422 1006	1.477 4554	1.593 8481	**8**
9	1.423 3118	1.486 0951	1.551 3282	1.689 4790	**9**
10	1.480 2443	1.552 9694	1.628 8946	1.790 8477	**10**
11	1.539 4541	1.622 8530	1.710 3394	1.898 2986	**11**
12	1.601 0322	1.695 8814	1.795 8563	2.012 1965	**12**
13	1.665 0735	1.772 1961	1.885 6491	2.132 9283	**13**
14	1.731 6764	1.851 9449	1.979 9316	2.260 9040	**14**
15	1.800 9435	1.935 2824	2.078 9282	2.396 5582	**15**
16	1,872 9812	2.022 3702	2.182 8746	2.540 3517	**16**
17	1.947 9005	2.113 3768	2.292 0183	2.692 7728	**17**
18	2.025 8165	2.208 4788	2.406 6192	2.854 3392	**18**
19	2.106 8492	2.307 8603	2.526 9502	3.025 5995	**19**
20	2.191 1231	2.411 7140	2.653 2977	3.207 1355	**20**

317. Remarks on the Table of Roots and Powers.

1. By the arithmetic process for finding square roots we find that $\sqrt{11} = 3.3166 \cdots$. If we had stopped the work at 3.316, we should have said that we had found the first three decimals of the root, or the root *to three decimal places*. However, the number 3.3166 is nearer to 3.317 than it is to 3.316. Hence we say that 3.317 is the root *to the nearest thousandth.*

In previous grades the pupil has found roots to a certain number of decimal places. *When, however, numbers taken from several different tables are to be used in a single problem* (as in ex. 17 to 20, page 236), *all the tables must have the same degree of accuracy.* Hence the table on page 271 gives roots to the nearest thousandth.

2. Since $\sqrt{99} = \sqrt{9}\sqrt{11} = 3\sqrt{11}$, we might expect that $\sqrt{99} = 3(3.317) = 9.951$. In the table, however, $\sqrt{99} = 9.950$. Actually $\sqrt{99} = 9.9498744 \cdots$. The difference is due to the fact that in multiplying by 3 we multiply any error in the table by 3. Hence, *when using a table* it is advisable to have the *coefficient* of the radical as small as possible. In algebraic work, on the other hand, a radical like $\sqrt{8x^5}$ is not considered in its simplest form unless the *radicand* is as small as possible.

3. When finding the value of $\dfrac{6+\sqrt{76}}{10}$ or any expression in which a root is combined with other numbers, the arithmetic work should always be carried out one step farther than the last figure we wish to retain, and the result then expressed to the nearest thousandth. Thus:

$$\frac{6+\sqrt{76}}{10} = \frac{6+8.718}{10} = \frac{14.718}{10} = 1.4718,$$

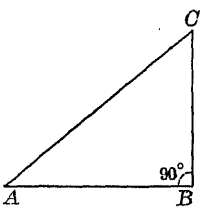
which, to the nearest thousandth, is 1.472.

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
1	1	1	1.000	1.000	**51**	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	**52**	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	**53**	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	**54**	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	**55**	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	**56**	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	**57**	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	**58**	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	**59**	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	**61**	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	**62**	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	**63**	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	**64**	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	**65**	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	**66**	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	**67**	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	**68**	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	**69**	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	**71**	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	**72**	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	**73**	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	**74**	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	**76**	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	**77**	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	**78**	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	**79**	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	**81**	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	**82**	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	**83**	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	**84**	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	**85**	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	**86**	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	**87**	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	**88**	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	**89**	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	**90**	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	**91**	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	**92**	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	**93**	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	**94**	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	**95**	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	**96**	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	**97**	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	**98**	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	**99**	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.642

318. Explanation of the Table of Trigonometric Ratios. To find the sine, the cosine, or the tangent of an angle between 0° and 45°, look for the angle in the *left*-hand columns, and find the name of the ratio at the *top* of the columns. The sine is found in the first column, the tangent in the second column, and the cosine in the fourth column.

The column marked " cot " gives the *cotangent* of the angle. The cotangent of an angle is defined as the reciprocal of the tangent; that is, the cotangent of A is 1 divided by the tangent of A. Hence, whenever we wish to divide by a tangent, we may, instead, multiply by the cotangent. The cotangent may also be defined as the ratio of the adjacent side to the opposite side of the right triangle.

For angles larger than 45°, look for the angle in the *right*-hand columns and find the name of the ratio at the *bottom* of the column instead of at the top. Thus, we see that sin 48° 20′ is the same number, .7470, as cos 41° 40′, and cos 48° 20′ is the same number, .6648, as sin 41° 40′. Similarly, the tangent of one angle is the cotangent of the other. Why this is true can be seen by examining the definitions of the ratios:



$$\sin A = \frac{BC}{AC} \qquad \cos C = \frac{BC}{AC}$$

$$\cos A = \frac{AB}{AC} \qquad \sin C = \frac{AB}{AC}$$

$$\tan A = \frac{BC}{AB} \qquad \cot C = \frac{BC}{AB}$$

$$\cot A = \frac{AB}{BC} \qquad \tan C = \frac{AB}{BC}$$

The sine of an angle equals the cosine of its complement.

The cosine of an angle equals the sine of its complement.

The tangent (or cotangent) of an angle equals the cotangent (or tangent) of its complement.

∠	sin	tan	cot	cos	
0°	.0000	.0000		[illegible]	**90°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
1°	.0175	.0175	[illegible]	.9998	**89°**
10′	[illegible]	[illegible]	[illegible]	.9998	50′
20′	[illegible]	[illegible]	[illegible]	.9997	40′
30′	[illegible]	[illegible]	[illegible]	.9997	30′
40′	[illegible]	[illegible]	[illegible]	.9996	20′
50′	[illegible]	[illegible]	[illegible]	.9995	10′
2°	[illegible]	[illegible]	[illegible]	.9994	**88°**
10′	[illegible]	[illegible]	[illegible]	.9993	50′
20′	[illegible]	[illegible]	[illegible]	.9992	40′
30′	[illegible]	[illegible]	[illegible]	.9990	30′
40′	[illegible]	[illegible]	[illegible]	.9989	20′
50′	[illegible]	[illegible]	[illegible]	.9988	10′
3°	[illegible]	[illegible]	[illegible]	.9986	**87°**
10′	[illegible]	[illegible]	[illegible]	.9985	50′
20′	[illegible]	[illegible]	[illegible]	.9983	40′
30′	[illegible]	[illegible]	[illegible]	.9981	30′
40′	[illegible]	[illegible]	[illegible]	.9980	20′
50′	[illegible]	[illegible]	[illegible]	.9978	10′
4°	[illegible]	[illegible]	[illegible]	.9976	**86°**
10′	[illegible]	[illegible]	[illegible]	.9974	50′
20′	[illegible]	[illegible]	[illegible]	.9971	40′
30′	[illegible]	[illegible]	[illegible]	.9969	30′
40′	[illegible]	[illegible]	[illegible]	.9967	20′
50′	[illegible]	[illegible]	[illegible]	.9964	10′
5°	.0872	.0875	11.430	.9962	**85°**
10′	.0901	.0904	11.059	.9959	50′
20′	.0929	.0934	10.712	.9957	40′
30′	[illegible]	[illegible]	[illegible]	.9954	30′
40′	[illegible]	[illegible]	[illegible]	.9951	20′
50′	[illegible]	[illegible]	[illegible]	.9948	10′
6°	.1045	.1051	9.5144	.9945	**84°**
10′	.1074	.1080	9.2553	.9942	50′
20′	.1103	.1110	9.0098	.9939	40′
30′	.1132	.1139	8.7769	.9936	30′
40′	.1161	.1169	8.5555	.9932	20′
50′	.1190	.1198	8.3450	.9929	10′
7°	.1219	.1228	8.1443	.9925	**83°**
10′	.1248	.1257	7.9530	.9922	50′
20′	.1276	.1287	7.7704	.9918	40′
30′	.1305	.1317	7.5958	.9914	30′
	cos	cot	tan	sin	∠

∠	sin	tan	cot	cos	
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
8°	[illegible]	[illegible]	[illegible]	[illegible]	**82°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.1478	.1495	6.6912	.9890	30′
40′	.1507	.1524	6.5606	.9886	20′
50′	.1536	.1554	6.4348	.9881	10′
9°	[illegible]	[illegible]	[illegible]	[illegible]	**81°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
10°	.1736	.1763	5.6713	.9848	**80°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.1822	.1853	5.3955	.9833	30′
40′	.1851	.1883	5.3093	.9827	20′
50′	.1880	.1914	5.2257	.9822	10′
11°	[illegible]	[illegible]	[illegible]	[illegible]	**79°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
12°	[illegible]	[illegible]	[illegible]	[illegible]	**78°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
13°	[illegible]	[illegible]	[illegible]	[illegible]	**77°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.2334	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
14°	[illegible]	[illegible]	[illegible]	[illegible]	**76°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
15°	.2588	.2679	3.7321	.9659	**75°**
	cos	cot	tan	sin	∠

∠	sin	tan	cot	cos	
15°	.2588	.2679	3.7321	.9659	**75°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.2672	.2773	3.6059	.9636	30′
40′	.2700	.2805	3.5656	.9628	20′
50′	.2728	.2836	3.5261	.9621	10′
16°	.2756	.2867	3.4874	.9613	**74°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.2840	.2962	3.3759	.9588	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
17°	.2924	.3057	3.2709	.9563	**73°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3007	.3153	3.1716	.9537	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
18°	.3090	.3249	3.0777	.9511	**72°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3173	.3346	2.9887	.9483	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
19°	.3256	.3443	2.9042	.9455	**71°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3338	.3541	2.8239	.9426	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
20°	[illegible]	[illegible]	[illegible]	[illegible]	**70°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3502	.3739	2.6746	.9367	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
21°	.3584	.3839	2.6051	.9336	**69°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3665	.3939	2.5386	.9304	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
22°	.3746	.4040	2.4751	.9272	**68°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3827	.4142	2.4142	.9239	30′
	cos	cot	tan	sin	∠

∠	sin	tan	cot	cos	
30′	.3827	.4142	2.4142	.9239	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
23°	[illegible]	[illegible]	[illegible]	[illegible]	**67°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.3987	.4348	2.2998	.9171	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
24°	[illegible]	[illegible]	[illegible]	[illegible]	**66°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.4147	.4557	2.1943	.9100	30′
40′	.4173	.4592	2.1775	.9088	20′
50′	.4200	.4628	2.1609	.9075	10′
25°	[illegible]	[illegible]	[illegible]	[illegible]	**65°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.4305	.4770	2.0965	.9026	30′
40′	.4331	.4806	2.0809	.9013	20′
50′	.4358	.4841	2.0655	.9001	10′
26°	.4384	.4877	2.0503	.8988	**64°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
27°	[illegible]	[illegible]	[illegible]	[illegible]	**63°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.4617	.5206	1.9210	.8870	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
28°	[illegible]	[illegible]	[illegible]	[illegible]	**62°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.4772	.5430	1.8418	.8788	30′
40′	.4797	.5467	1.8291	.8774	20′
50′	.4823	.5505	1.8165	.8760	10′
29°	[illegible]	[illegible]	[illegible]	[illegible]	**61°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	.4924	.5658	1.7675	.8704	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
30°	.5000	.5774	1.7321	.8660	**60°**
	cos	cot	tan	sin	∠

∠	sin	tan	cot	cos	
30°	.5000	.5774	[illegible]	[illegible]	**60°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.5075	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
31°	[illegible]	[illegible]	[illegible]	[illegible]	**59°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.5225	.6128	1.6319	.8526	30′
40′	.5250	.6168	1.6212	.8511	20′
50′	.5275	.6208	1.6107	.8496	10′
32°	[illegible]	[illegible]	[illegible]	[illegible]	**58°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.5373	.6371	1.5697	.8434	30′
40′	.5398	.6412	1.5597	.8418	20′
50′	.5422	.6453	1.5497	.8403	10′
33°	[illegible]	[illegible]	[illegible]	[illegible]	**57°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.5519	.6619	1.5108	.8339	30′
40′	.5544	.6661	1.5013	.8323	20′
50′	.5568	.6703	1.4919	.8307	10′
34°	[illegible]	[illegible]	[illegible]	[illegible]	**56°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
35°	.5736	.7002	1.4281	.8192	**55°**
10′	.5760	.7046	1.4193	.8175	50′
20′	.5783	.7089	1.4106	.8158	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
36°	[illegible]	[illegible]	[illegible]	[illegible]	**54°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.5948	[illegible]	[illegible]	[illegible]	30
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
37°	[illegible]	[illegible]	[illegible]	[illegible]	**53°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.6088	.7673	1.3032	.7934	30′
	cos	cot	tan	sin	∠

∠	sin	tan	cot	cos	
30′	.6088	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
38°	[illegible]	[illegible]	[illegible]	[illegible]	**52°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.6225	.7954	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
39°	.6293	.8098	1.2349	.7771	**51°**
10′	.6316	.8146	1.2276	.7753	50′
20′	.6338	.8195	1.2203	.7735	40′
30′	.6361	.8243	1.2131	.7716	30′
40′	.6383	.8292	1.2059	.7698	20′
50′	.6406	.8342	1.1988	.7679	10′
40°	[illegible]	[illegible]	[illegible]	[illegible]	**50°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	.6494	.8541	1.1708	.7604	30′
40′	.6517	.8591	1.1640	.7585	20′
50′	.6539	.8642	1.1571	.7566	10′
41°	.6561	.8693	1.1504	.7547	**49°**
10′	.6583	.8744	1.1436	.7528	50′
20′	.6604	.8796	1.1369	.7509	40′
30′	.6626	.8847	1.1303	.7490	30′
40′	.6648	.8899	1.1237	.7470	20′
50′	.6670	.8952	1.1171	.7451	10′
42°	[illegible]	[illegible]	[illegible]	[illegible]	**48°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
43°	[illegible]	[illegible]	[illegible]	[illegible]	**47°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
44°	[illegible]	[illegible]	[illegible]	[illegible]	**46°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
45°	.7071	1.000	1.0000	.7071	**45°**
	cos	cot	tan	sin	∠

319. Explanation of the Table of Logarithms of Trigonometric Ratios. The explanation on page 272 must be read first.

The table on pages 277 to 279 gives the logarithms of the numbers that are found on pages 273 to 275; that is, the preceding table gives the actual values of the ratios, while the following table gives the logarithms of the ratios. The two tables are sometimes distinguished by the names "natural" ratios and "logarithmic" ratios.

As in the preceding table:

For angles less than 45°, look for the angle in the *left*-hand columns and find the name of the ratio at the *top* of the columns.

For angles greater than 45°, look for the angle in the *right*-hand columns and find the name of the ratio at the *bottom* of the columns.

Since all sines and cosines are smaller than 1, their logarithms have negative characteristics. In the table, however, the logarithm shows a positive characteristic and -10 is understood with each such logarithm.

Likewise, if an angle is less than 45°, its tangent is less than 1, and hence its logarithm has a negative characteristic. Again -10 is understood with each such logarithm.

If, however, an angle is greater than 45°, then its tangent is greater than 1, and the logarithm has either the characteristic 0 or some positive characteristic, which is printed in the table.

∠	log sin	log tan	log cot	log cos	
0°	[illegible]	[illegible]	[illegible]	[illegible]	90°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
1°	[illegible]	[illegible]	[illegible]	[illegible]	89°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
2°	[illegible]	[illegible]	[illegible]	[illegible]	88°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
3°	[illegible]	[illegible]	[illegible]	[illegible]	87°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
4°	[illegible]	[illegible]	[illegible]	[illegible]	86°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
5°	[illegible]	[illegible]	[illegible]	[illegible]	85°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
6°	[illegible]	[illegible]	[illegible]	[illegible]	84°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
7°	[illegible]	[illegible]	[illegible]	[illegible]	83°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.1157	9.1194	0.8806	9.9963	30′
	log cos	log cot	log tan	log sin	∠

∠	log sin	log tan	log cot	log cos	
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
8°	9.1436	9.1478	0.8522	9.9958	82°
10′	9.1525	9.1569	0.8431	9.9956	50′
20′	9.1612	9.1658	0.8342	9.9954	40′
30′	9.1697	9.1745	0.8255	9.9952	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
9°	9.1943	9.1997	0.8003	9.9946	81°
10′	9.2022	9.2078	0.7922	9.9944	50′
20′	9.2100	9.2158	0.7842	9.9942	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
10°	[illegible]	[illegible]	[illegible]	[illegible]	80°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
11°	[illegible]	[illegible]	[illegible]	[illegible]	79°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
12°	[illegible]	[illegible]	[illegible]	[illegible]	78°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
13°	[illegible]	[illegible]	[illegible]	[illegible]	77°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
14°	[illegible]	[illegible]	[illegible]	[illegible]	76°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
15°	9.4130	9.4281	0.5719	9.9849	75°
	log cos	log cot	log tan	log sin	∠

∠	log sin	log tan	log cot	log cos	
15°	9.4130	9.4281	0.5719	9.9849	**75°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.4269	9.4430	0.5570	9.9839	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
16°	9.4403	9.4575	0.5425	9.9828	**74°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.4533	9.4716	0.5284	9.9817	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
17°	9.4659	9.4853	0.5147	9.9806	**73°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
18°	9.4900	9.5118	0.4882	9.9782	**72°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
19°	[illegible]	[illegible]	[illegible]	[illegible]	**71°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
20°	[illegible]	[illegible]	[illegible]	[illegible]	**70°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
21°	9.5543	9.5842	0.4158	9.9702	**69°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	0.4046	9.9687	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
22°	9.5736	9.6064	0.3936	9.9672	**68°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	9.5828	9.6172	0.3828	9.9656	30′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
23°	[illegible]	[illegible]	[illegible]	[illegible]	**67°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
24°	[illegible]	[illegible]	[illegible]	[illegible]	**66°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.6177	9.6587	0.3413	9.9590	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
25°	9.6259	9.6687	0.3313	9.9573	**65°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.6340	9.6785	0.3215	9.9555	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
26°	[illegible]	[illegible]	[illegible]	[illegible]	**64°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
27°	9.6570	9.7072	0.2928	9.9499	**63°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.6644	9.7165	0.2835	9.9479	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
28°	9.6716	9.7257	0.2743	9.9459	**62°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.6787	9.7348	0.2652	9.9439	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
29°	9.6856	9.7438	0.2562	9.9418	**61°**
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.6923	9.7526	0.2474	9.9397	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
30°	9.6990	9.7614	0.2386	9.9375	**60°**
	log cos	log cot	log tan	log sin	∠

∠	log sin	log tan	log cot	log cos	
30°	9.6990	9.7614	[illegible]	[illegible]	60°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	9.7055	9.7701	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
31°	[illegible]	[illegible]	[illegible]	[illegible]	59°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	9.7181	9.7873	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
32°	[illegible]	[illegible]	[illegible]	[illegible]	58°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
33°	[illegible]	[illegible]	[illegible]	[illegible]	57°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
34°	9.7476	9.8290	0.1710	[illegible]	56°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.7531	9.8371	0.1629	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
35°	[illegible]	[illegible]	[illegible]	[illegible]	55°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	9.7640	9.8533	0.1467	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
36°	[illegible]	[illegible]	[illegible]	[illegible]	54°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.7744	9.8692	0.1308	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
37°	[illegible]	[illegible]	[illegible]	[illegible]	53°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	9.7844	9.8850	0.1150	9.8995	30′
	log cos	log cot	log tan	log sin	∠

∠	log sin	log tan	log cot	log cos	∠
30′	9.7844	9.8850	0.1150	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
38°	[illegible]	[illegible]	[illegible]	[illegible]	52°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
39°	[illegible]	[illegible]	[illegible]	[illegible]	51°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
40°	[illegible]	[illegible]	[illegible]	[illegible]	50°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
41°	[illegible]	[illegible]	[illegible]	[illegible]	49°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	9.8213	9.9468	0.0532	9.8745	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
42°	[illegible]	[illegible]	[illegible]	[illegible]	48°
10′	[illegible]	[illegible]	[illegible]	[illegible]	50′
20′	[illegible]	[illegible]	[illegible]	[illegible]	40′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
43°	[illegible]	[illegible]	[illegible]	[illegible]	47°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
44°	[illegible]	[illegible]	[illegible]	[illegible]	46°
10′	[illegible]	[illegible]	[illegible]	[illegible]	0′
20′	[illegible]	[illegible]	[illegible]	[illegible]	0′
30′	[illegible]	[illegible]	[illegible]	[illegible]	30′
40′	[illegible]	[illegible]	[illegible]	[illegible]	20′
50′	[illegible]	[illegible]	[illegible]	[illegible]	10′
45°	9.8495	0.0000	0.0000	9.8495	45°
	log cos	log cot	log tan	log sin	∠

log π = .4971

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8616	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

FORMULAS

$A = \sqrt{s(s-a)(s-b)(s-c)}$, $s = \frac{1}{2}(a+b+c)$	Area of a triangle in terms of its sides
$F = \frac{9}{5}C + 32$	Fahrenheit and centigrade temperatures
$C = 2\pi r$	Circumference of a circle
$A = \pi r^2$	Area of a circle.
$A = 4\pi r^2$	Area of the surface of a sphere
$V = \frac{4}{3}\pi r^3$	Volume of a sphere
$V = \frac{1}{3}bh$	Volume of a pyramid
$V = \pi r^2 h$	Volume of a cylinder
$V = 2\pi r^2 + 2\pi rh$	Total area of surface of a cylinder
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Roots of $ax^2 + bx + c = 0$
$r_1 + r_2 = -\dfrac{b}{a}$; $r_1 r_2 = \dfrac{c}{a}$	Sum and product of the roots
$l = a + (n-1)d$, $S = \dfrac{n}{2}(a+l)$	The nth term and the sum of n terms of an arithmetic progression
l or $t_n = ar^{n-1}$, $S_n = a\dfrac{r^n - 1}{r - 1}$	The nth term and the sum of n terms of a geometric progression
$S = \dfrac{a}{1-r}$	The sum of an infinite geometric progression
$A = P(1+i)^n$	Amount on P dollars at $i\%$ compounded annually for n years.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

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